Vertex operator algebras associated to admissible representations of $\hat{sl}_2$

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1   Introduction

Let $\{e, f, h\}$ be a standard basis for $\mathfrak{g} = sl_2$ such that $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$, $\hat{\mathfrak{g}}$ the corresponding affine Lie algebra and $L(\ell, j)$ the irreducible highest weight $\hat{\mathfrak{g}}$-module of level ℓ with highest weight j. It is well known that the vacuum representation $L(\ell, 0)$ has a natural vertex operator algebra (or chiral algebra) structure for any $\ell \neq -2$ (cf. [FZ]). If ℓ is a positive integer, the chiral algebra $L(\ell, 0)$ of the WZNW models in the content of conformal field theory has been well understood. For example, the fusion rules are obtained by using primary field decomposition (cf. [GW], [TK]) or by Verlinde formula (cf. [K], [V]), n point functions are calculated [KZ].

In the content of vertex operator algebra, it has been proved (cf. [DL], [FL], [Li1]) that any $\mathbb{Z}_+$-graded weak $L(\ell, 0)$-module is completely reducible and the set of equivalence classes of irreducible $L(\ell, 0)$-modules is the set of equivalence classes of standard $\hat{\mathfrak{g}}$-modules of level ℓ. Thus $L(\ell, 0)$ is rational (defined in Section 2). (It has been proved recently in [DLiM2] that *any weak $L(\ell, 0)$-module* is completely reducible and the set of equivalence classes of irreducible *weak $L(\ell, 0)$-modules* is the set of equivalence classes of standard $\hat{\mathfrak{g}}$-modules of level ℓ.) The modular invariance of the vector space linearly spanned by the characters $tr_{L(\ell, j)} e^{2\pi i \tau (L(0) - \frac{c_\ell}{24})}$ for all standard modules of level ℓ is obtained in [KP] by using the explicit character formulas or follows from a general theorem of Zhu [Z].

The fusion rules are computed in [FZ] by studying certain associative algebras and its bimodules associated to $L(\ell, 0)$ and its irreducible modules.

If ℓ is rational such that $\ell + 2 = \dfrac{p}{q}$ for some coprime positive integers $p \geq 2$ and q, Kac and Wakimoto [KW1]-[KW2] found finitely many distinguished irreducible representations, called admissible (or modular invariant) representations. In this case the fusion rules among admissible modules have been calculated in the content of conformal field theory (cf. [AY], [BF], [MW]) by employing different methods, but different methods sometime give different results. Especially, Verlinde formula gives negative fusion rules.

If j is not an integer, $\mathrm{tr}_{L(\ell,j)} e^{2\pi i \tau (L(0) - \frac{c_\ell}{24})}$ does not exist (because the homogeneous subspaces are infinite-dimensional) so that the character $\mathrm{tr}_{L(\ell,j)} e^{2\pi i \tau (L(0) - \frac{1}{2}zh(0) - \frac{c_\ell}{24})}$ was considered in [KW1]-[KW2], where z is a positive rational number less than 1. In [KW1], a formula for $\mathrm{tr}_{L(\ell,j)} e^{2\pi i \tau (L(0) - \frac{1}{2}zh(0) - \frac{c_\ell}{24})}$ in terms of theta functions was given and a transformation law under $S(\tau, z) = (-\tau^{-1}, -z\tau)$ were found. Later, the transformation law was corrected by adding an extra factor [KW2]. After this correction, it is not clear that the space linearly spanned by all $\mathrm{tr}_{L(\ell,j)} e^{2\pi i \tau (L(0) - \frac{1}{2}zh(0) - \frac{c_\ell}{24})}$ for admissible weights j is invariant under the action of the modular group $PSL(2, \mathbb{Z})$, where $S(\tau, z) = (-\tau^{-1}, -z\tau)$.

Our modest purpose of this paper is to study these admissible representations from the point of view of vertex operator algebra. We show that $L(\ell, 0)$ is a $\mathbb{Q}$-graded rational vertex operator algebra under a new Virasoro algebra and its irreducible modules are exactly these admissible modules for $\hat{\mathfrak{g}}$. We extend Zhu's $A(V)$-theory ([FZ],[Z]) to $\mathbb{Q}$-graded vertex operator algebras and apply this theory to $L(\ell, 0)$ to calculate the fusion rules. The new Virasoro algebra also gives a natural interpretation of the characters $\mathrm{tr}_{L(\ell,j)} e^{2\pi i \tau (L(0) - \frac{1}{2}zh(0) - \frac{c_\ell}{24})}$.

We explain these results in details in the following. In the first part of the paper, we prove that all the admissible representations of level ℓ constitute the set of irreducible $\mathbb{Z}_+$-graded weak $L(\ell, 0)$-modules among all the highest-weight irreducible $\hat{\mathfrak{g}}$-modules of

level ℓ. This has been implicit in references such as [AY], [BF], [FM]. It follows from this result and a complete reducibility theorem of Kac-Wakimoto [KW2] that any weak $L(\ell, 0)$-module from category $\mathcal{O}$ is completely reducible. Let N_+ be the sum of all positive root spaces of $\hat{\mathfrak{g}}$. Let $\mathcal{E}$ be the category of weak $L(\ell, 0)$-modules W on which N_+ is locally nilpotent, i.e., for any $u \in W$, there is a positive integer k such that $N_+^k u = 0$. Then we prove that any weak $L(\ell, 0)$-module from category $\mathcal{E}$ is completely reducible.

Since some admissible weights are not integers if ℓ is not integral, Zhu's algebra $A(L(l, 0))$ [Z] has infinite-dimensional irreducible modules. This implies that $L(\ell, 0)$ is not rational and that Zhu's C_2-condition (another crucial condition for Zhu's theorem of modular invariance) is not true either.

In the second part of the paper, we study the vertex operator algebra $L(\ell, 0)$ under a new Virasoro algebra. Let ω be the original Segal-Sugawara Virasoro vector of $L(\ell, 0)$. Set $\omega_z = \omega + \frac{1}{2}zh(-2)\mathbf{1} \in L(\ell, 0)$, where z is a complex number. Then ω_z is a Virasoro vector of $L(\ell, 0)$ with a central charge $c_{\ell, z} = c_\ell - 6\ell z^2$ and $L_z(0) = (\omega_z)_1 = L(0) - \frac{1}{2}zh(0)$. If we choose $z = 0, \frac{1}{2}$, we obtain the homogeneous grading and the rescaled principal grading (cf. [K], [LW]), respectively. Let z be positive rational number less than 1. Note that the vertex operator algebra $(L(\ell, 0), Y, \mathbf{1}, \omega_z)$ is $\mathbb{Q}$-graded instead of $\mathbb{Z}$-graded. We extend Zhu's $A(V)$-theory of one-to-one correspondence [Z] between the set of equivalence classes of irreducible admissible V-modules and the set of equivalence classes of irreducible $A(V)$-modules and Frenkel-Zhu's $A(M)$ theory [FZ] for fusion rules to any $\mathbb{Q}$-graded vertex operator algebra. It follows from our complete reducibility theorem in the first part that any $\mathbb{Q}_+$-graded weak $L(\ell, 0)$-module under the new Virasoro vector ω_z is completely reducible. That is, $(L(\ell, 0), \omega_z)$ is rational. By using the Malikov-Feigin-Fuchs's singular vector expressions [MFF] and the Fuchs' projection formula [F] we find all the fusion rules and prove that $(L(\ell, 0), \omega_z)$ satisfies the C_2-finite condition. Our results on fusion rule agree with the corresponding results in [BF].

It is natural for us to consider the modified characters $\mathrm{tr}e^{2\pi i\tau(L_z(0)-\frac{1}{24}c_{\ell,z})}$, that is, $\mathrm{tr}e^{2\pi i\tau(L(0)-\frac{1}{2}zh(0)-\frac{1}{24}(c_\ell-6\ell z^2))}$. Using KW's character formula [KW1] we find that these modified characters are modular functions so that $c_{\ell,z}$ is the modular anomaly rather than c_ℓ. (Then the characters $\mathrm{tr}e^{2\pi i\tau(L(0)-\frac{1}{2}zh(0)-\frac{1}{24}c_\ell)}$ are obviously not modular functions.) One may ask: Is the space linearly spanned by our new characters invariant under the transformation $S(\tau) = -\tau^{-1}$ with z being fixed? This will be discussed in our coming paper [DLiM3].

We should mention that the vertex operator algebras associated to irreducible highest weight representations of certain rational levels for affine Lie algebra $C_n^{(1)}$ have been studied in [A].

2   Vertex operator algebras $L(\ell, 0)$ associated to $\hat{sl}_2$

A vertex operator algebra, or briefly a VOA, is a $\mathbb{Z}$-graded vector space $V = \oplus_{n\in\mathbb{Z}}V_n$ satisfying a number of axioms. We refer the reader to [B], [FLM] and [FHL] for the details of the definition. However, we would like to give the definitions of weak modules and modules in details. A *weak V-module* is a pair (W, Y_W), where W is a vector space and $Y_W(\cdot, z)$ is a linear map from V to $(\mathrm{End}W)[[z, z^{-1}]]$ satisfying the following axioms: (1) $Y_W(\mathbf{1}, z) = id_W$; (2) $Y_W(a, z)u \in W((z))$ for any $a \in V, u \in W$; (3) $Y_W(L(-1)a, z) = \frac{d}{dz}Y_W(a, z)$ for $a \in V$; (4) the Jacobi identity:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_M(a, z_1)Y_M(b, z_2)u - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)Y_M(b, z_2)Y_M(a, z_1)u$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y_M(Y(a, z_0)b, z_2)u \tag{2.1}$$

for $a, b \in V, u \in W$. A weak V-module (W, Y_W) is called a *V-module* if $L(0)$ semisimply acts on W with the decomposition into $L(0)$-eigenspaces $M = \oplus_{h\in\mathbb{C}}M_h$ such that for any $h \in \mathbb{C}$, $\dim M_h < \infty$, $M_{h+n} = 0$ for $n \in \mathbb{Z}$ sufficiently small.

A $\mathbb{Z}_+$-*graded weak V*-module [FZ] is a weak V-module W together with a $\mathbb{Z}_+$-gradation

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$W = \oplus_{n=0}^{\infty} W(n)$ such that

$$a_m W(n) \subseteq W(k + n - m - 1) \quad \text{for } a \in V_k, m, n \in \mathbb{Z}, \tag{2.2}$$

where $W(n) = 0$ by definition for $n < 0$. One may define the notions of "submodule" and "irreducible submodule" accordingly. A VOA V is said to be *rational* if any $\mathbb{Z}_+$-graded weak V-module is a direct sum of irreducible $\mathbb{Z}_+$-graded weak V-modules. It was proved in [DLiM1] that if V is rational, there are only finitely many irreducible $\mathbb{Z}_+$-graded weak V-modules up to equivalence and any irreducible weak V-module is a module.

Let $\{e, f, h\}$ be the standard basis for $\mathfrak{g} = sl_2$ with the commutation relations: $[e, f] = h.[h, e] = 2e, [h, f] = -2f$. We fix the normalized Killing form on $\mathfrak{g}$ such that $\langle h, h \rangle = 2$. Let $\tilde{\mathfrak{g}} = \tilde{sl}_2 = \mathbb{C}[x, x^{-1}] \otimes \mathfrak{g} + \mathbb{C}c$ be the affine Lie algebra and identify $\mathfrak{g}$ with $x^0 \otimes \mathfrak{g}$. Set $a(n) = a \otimes x^n$ for $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$ for convenience. Define subalgebras

$$N_+ = \mathbb{C}e + x\mathbb{C}[x] \otimes \mathfrak{g}, \quad N_- = \mathbb{C}f + x^{-1}\mathbb{C}[x^{-1}] \otimes \mathfrak{g}, \tag{2.3}$$

$$B = N_+ \oplus \mathbb{C}h \oplus \mathbb{C}c, \quad P = \mathbb{C}[x] \otimes \mathfrak{g} \oplus \mathbb{C}c. \tag{2.4}$$

Then $\tilde{sl}_2 = N_+ \oplus \mathbb{C}h \oplus \mathbb{C}c \oplus N_-$.

Let $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}d$ be the extended affine algebra [K], where

$$[d, c] = 0, \quad [d, x^n \otimes a] = n(x^n \otimes a) \quad \text{for } a \in \mathfrak{g}, n \in \mathbb{Z}.$$

Let $H = \mathbb{C}h \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the Cartan subalgebra of $\hat{\mathfrak{g}}$, α_0, α_1 be the simple roots of $\hat{\mathfrak{g}}$, let $\Gamma_+ = \mathbb{Z}_+\alpha_0 \oplus \mathbb{Z}_+\alpha_1$ and let Λ_0, Λ_1 be the fundamental weights of $\hat{\mathfrak{g}}$ [K]. Let $\bar{\rho}$ be half of the sum of positive roots of $\mathfrak{g}$ and set $\rho = \bar{\rho} + 2\Lambda_0$ [K]. For any $\lambda \in H^*$, denote by $M(\lambda)$ (resp. $L(\lambda)$) the Verma (resp. the irreducible highest weight) $\hat{\mathfrak{g}}$-module. When restricted to $\tilde{\mathfrak{g}}$, $L(\lambda)$ is an irreducible $\tilde{\mathfrak{g}}$-module [K]. It is clear that $L(\lambda)$ and $L(\mu)$ are isomorphic $\tilde{\mathfrak{g}}$-module if and only if $\lambda \in \mu + \mathbb{C}\delta$. As commonly used in many references, we use the notation $L(\ell, j)$ for the $\tilde{\mathfrak{g}}$-module $L(\lambda)$, where $\ell = \langle \lambda, c \rangle, j = \langle \lambda, \alpha_1 \rangle = j$. Conversely, let M be a restricted $\tilde{\mathfrak{g}}$-module of level $\ell \neq -2$. Then we extend M to a $\hat{\mathfrak{g}}$-module by letting d

act on M as $-L(0)$. In this paper we shall consider any restricted $\tilde{\mathfrak{g}}$-module as a $\hat{\mathfrak{g}}$-module in this way.

For a complex number l and a $\mathbb{C}h$-module U which can be regarded as a B-module by N_+ acting trivially and c acting as l, let $M(\ell, U)$ be the *generalized Verma $\tilde{\mathfrak{g}}$-module* $U(\tilde{\mathfrak{g}}) \otimes_{U(B)} U$ [Le] of level l or Weyl module. If $U = \mathbb{C}$ is one-dimensional $\mathbb{C}h$-module on which h acts as a fixed complex number j the corresponding module is an ordinary *Verma module* denoted by $M(l, j)$. Note that U can be identify with the subspace $1 \otimes_{U(B)} U$ of $M(l, U)$. Then $M(l, j)$ has a unique maximal submodule which intersects trivially with $\mathbb{C}$ and $L(\ell, j)$ is isomorphic to the corresponding irreducible highest weight module.

Similarly, one can define the generalized Verma $\tilde{\mathfrak{g}}$-module $V(\ell, U) = U(\tilde{\mathfrak{g}}) \otimes_{U(P)} U$ for any $\mathfrak{g}$-module U which can be extended to a P-module by setting $x\mathbb{C}[x] \otimes \mathfrak{g}$ acting trivially and c acting as l. Note that if $U = \mathbb{C}$ is the trivial $\mathfrak{g}$-module then $V(\ell, \mathbb{C})$ is a quotient of $M(l, 0)$ and $L(l, 0)$ is the irreducible quotient of $V(\ell, \mathbb{C})$ modulo the unique maximal submodule which intersects $\mathbb{C}$ trivially.

It is well-known that $V(\ell, \mathbb{C})$ and $L(\ell, 0)$ have natural vertex operator algebra structures for any $\ell \neq -2$ and that any $M(\ell, U)$ is a weak module for vertex operator algebra $V(\ell, \mathbb{C})$ (cf. [FZ] and [Li1]).

We recall the following Kac-Kazhdan reducibility criterion [KK]:

Proposition 2.1 *The Verma module $M(\ell, j)$ is reducible if and only if there are some positive integers n, k such that one of the three conditions hold:*

$$(I) \ \ j = n - 1 - (k-1)t; \quad (II) \ \ j = -n + kt; \quad (III) \ \ \ell + 2 = 0, \qquad (2.5)$$

where $t = \ell + 2$.

Remark 2.2 *Since any restricted $\tilde{\mathfrak{g}}$-module of level ℓ is a weak $V(\ell, \mathbb{C})$-module ([FZ], [Li1]), $V(\ell, \mathbb{C})$ is always irrational. If $t = \ell + 2 \notin \mathbb{Q}_+$, then it follows from Proposition 2.1 that $V(\ell, \mathbb{C}) = L(\ell, 0)$. Therefore, $L(\ell, 0)$ is an irrational vertex operator algebra.*

Recall from [KW1] that a weight $\lambda \in H^*$ is said to be *admissible* if the following conditions hold:

(i) $\langle \lambda + \rho, \alpha \rangle > 0$ for all but finitely many positive roots α of $\hat{\mathfrak{g}}$;

(ii) $\langle \lambda + \rho, \alpha \rangle \notin \{-1, -2, \cdots\}$ for any positive root α of $\hat{\mathfrak{g}}$;

(iii) The set of positive roots α satisfying $\langle \lambda + \rho, \alpha \rangle \in \mathbb{Z}_+$ spans a 2-dimensional subspace of H^*.

A complex number ℓ is called *an admissible level* if there is an admissible weight λ such that $\langle \lambda, c \rangle = \ell$. It was proved in [KW1] that ℓ is an admissible level if and only if $\ell = -2 + \frac{p}{q}$, where p and q are coprime positive integers with $p \geq 2$ and j is an admissible weight of level ℓ if and only if $j = n - k\frac{p}{q}$ for some $n, k \in \mathbb{Z}_+, n \leq p - 2, k \leq q - 1$. From now on we will assume that $t = \ell + 2 = \frac{p}{q}$, where p and q are coprime positive integers with $p \geq 2$.

Remark 2.3 *Let $j = n - kt$ be an admissible weight. Then $j = -(p - n) + (q - k)t$. Since p and q are relatively prime, $r - st = 0$ for $r, s \in \mathbb{Z}, 0 \leq s \leq q - 1$ if and only if $s = 0, r = 0$. Consequently, the expression $j = n - kt$ of an admissible weight j with $n, k \in \mathbb{Z}_+, n \leq q - 2, k \leq q - 1$ is unique.*

A vector w in a highest weight module M for $\tilde{\mathfrak{g}}$ is called a *singular vector* if w is a highest weight vector which generates a proper submodule. It is well known that the singular vectors of $M(\ell, j)$ give the key information for determining the module structure of $L(l, j)$ and the fusion rules. In [MFF] an expression for singular vectors in terms of non-integral powers of elements of $\hat{\mathfrak{g}}$ was found as follows (see [MFF] for details):

Proposition 2.4 [MFF] *Let $j = n - 1 - (k - 1)t$ where n and k are positive integers satisfying $1 \leq n \leq p - 1, 1 \leq k \leq q$ and let v be a highest weight vector of the Verma module $M(\ell, j)$. Set*

$$F_1(n, k) = f(0)^{n+(k-1)t} e(-1)^{n+(k-2)t} f(0)^{n+(k-3)t} e(-1)^{n+(k-4)t}$$

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$$\cdots e(-1)^{n-(k-2)t} f(0)^{n-(k-1)t}, \tag{2.6}$$

$$F_2(n,k) = e(-1)^{p-n+(q-k)t} f(0)^{p-n+(q-k-1)t} e(-1)^{p-n+(q-k-2)t} f(0)^{p-n+(q-k-3)t}$$

$$\cdots f(0)^{p-n-(q-k+1)t} e(-1)^{p-n-(q-k)t}. \tag{2.7}$$

Then $v_{j,1} = F_1(n,k)v, v_{j,2} = F_2(n,k)v$ are singular vectors of $M(\ell,j)$ of degrees $n((k-1)\alpha_0 + k\alpha_1)$ and $(p-n)((q+1-k)\alpha_0 + (q-k)\alpha_1)$, respectively. Moreover, the maximal proper submodule of $M(\ell,j)$ is generated by $v_{j,1}$ and $v_{j,2}$.

Remark 2.5 Note that $v_{0,2} = F_2(1,1)\mathbf{1}$ generates the maximal proper submodule of $V(\ell,\mathbb{C})$.

For any complex number α, following [F] and [FM] we set $H_\alpha = fe - \alpha h - \alpha(\alpha+1)$. Then

$$H_\alpha H_\beta = H_\beta H_\alpha, \quad e^m H_\alpha = H_{\alpha-m} e^m, \quad f^m H_\alpha = H_{\alpha+m} f^m, \tag{2.8}$$

$$f^m e^m = H_0 H_1 \cdots H_{m-1}, \quad e^m f^m = H_{-1} H_{-2} \cdots H_{-m}, \tag{2.9}$$

$$h^m e^n = e^n(h+2n)^m, \quad h^m f^n = f^n(h-2n)^m \tag{2.10}$$

for any complex numbers α, β and for any positive integers m, n.

Let σ be the anti-automorphism of $U(\mathfrak{g})$ such that $\sigma(a) = -a$ for any $a \in \mathfrak{g}$. Then $\sigma(H_\alpha) = H_{-(\alpha+1)}$ for any complex number α. Let P_1 be the projection $\tilde{\mathfrak{g}}$ onto $\mathfrak{g}$ such that $P_1(t^n \otimes a) = a$ for any $a \in \mathfrak{g}$ and $P_1(c) = 0$.

Proposition 2.6 *[F] The following projection formulas hold:*

$$P_1(F_1(n,k)) = \left(\prod_{r=0}^{n-1} \prod_{s=1}^{k-1} H_{r+st} \right) f^n, \tag{2.11}$$

$$P_1(F_2(n,k)) = \left(\prod_{r=1}^{p-n} \prod_{s=1}^{q-k} H_{-r-st} \right) e^{p-n}. \tag{2.12}$$

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Let $B_0 = \mathbb{C}(f(-1)+f(0))+\mathbb{C}[x^{-1}](x^{-2}+x^{-1})\otimes\mathfrak{g}$. Then B_0 is an ideal of N_- such that $N_-/B_0 = \mathbb{C}T_+ + \mathbb{C}T_0 + \mathbb{C}T_-$, denoted by L_0, where $T_+ = e(-1)+B_0, T_0 = h(-1)+B_0, T_- = f + B_0$, satisfies the following commutation relations:

$$[T_+,T_-] = T_0, \quad [T_0,T_+] = -2T_+, \quad [T_0,T_-] = 2T_-. \tag{2.13}$$

Let P be the natural quotient map from $U(N_-)$ onto $U(L_0)$. For any complex number α, we define $G_\alpha = T_-T_+ - \alpha T_0 + \alpha(\alpha+1)$. Then

$$G_\alpha G_\beta = G_\beta G_\alpha, \quad T_+^m G_\alpha = G_{\alpha-m} T_+^m, \quad T_-^m G_\alpha = G_{\alpha+m} T_-^m, \tag{2.14}$$

$$T_-^m T_+^m = G_0 G_1 \cdots G_{m-1}, \quad T_+^m T_-^m = G_{-1}G_{-2}\cdots G_{-m} \tag{2.15}$$

for any complex numbers α, β and for any positive integer m. Using the same method as suggested in [F] we obtain

Proposition 2.7 *The following formulas hold:*

$$P(F_1(n,k)) = \left(\prod_{r=0}^{n-1}\prod_{s=1}^{k-1} G_{r+st}\right) T_-^n, \tag{2.16}$$

$$P(F_2(n,k)) = \left(\prod_{r=1}^{p-n}\prod_{s=1}^{q-k} G_{-r-st}\right) T_+^{p-n}. \tag{2.17}$$

Recall that $N_- = \mathbb{C}f + x^{-1}\mathbb{C}[x^{-1}] \otimes \mathfrak{g}$. Set $B_2 = \mathbb{C}f(-1) + x^{-2}\mathbb{C}[x^{-1}] \otimes \mathfrak{g}$. The it is clear that B_2 is an ideal of N_-. Let $L_2 = N_-/B_2$ be the quotient Lie algebra. Then L_2 is a three-dimensional Heisenberg Lie algebra with relations: $[\bar{e}, \bar{f}] = \bar{h}, [\bar{h}, \bar{e}] = [\bar{h}, \bar{f}] = 0$, where $\bar{e} = e(-1) + B_2, \bar{f} = f + B_2, \bar{h} = h(-1) + B_2$. Let P_2 be the natural quotient map from $U(N_-)$ to $U(L_2)$. Then

Proposition 2.8 *[F] For any positive integers $1 \leq n \leq p-1, 1 \leq k \leq q$, we have*

$$P_2(F_1(n,k)) = \left(\prod_{r=0}^{n-1}\prod_{s=1}^{k-1} \bar{H}_{r+st}\right) \bar{f}^n, \tag{2.18}$$

$$P_2(F_2(n,k)) = \left(\prod_{r=1}^{p-n}\prod_{s=1}^{q-k} \bar{H}_{-r-st}\right) \bar{e}^{p-n}, \tag{2.19}$$

where $\bar{H}_\alpha = \bar{e}\bar{f} - \alpha\bar{h}$.

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For a $\mathbb{C}h$-module define a linear functional on $U^* \otimes M(\ell, U)$ as follows:

$$\langle u', u \rangle = u'(\mathcal{P}(u)) \quad \text{for } u' \in U^*, u \in M(\ell, U), \tag{2.20}$$

where $\mathcal{P}$ is the projection of $M(\ell, U)$ onto the subspace U. Define

$$I = \{u \in M(\ell, U) | \langle u', xu \rangle = 0 \quad \text{for any } u' \in U^*, x \in U(\tilde{\mathfrak{g}})\}. \tag{2.21}$$

It is clear that I is the unique maximal submodule which intersects with U trivially. Set $L(\ell, U) = M(\ell, U)/I$ and regard U as a subspace in a natural way. Then $\mathcal{P}$ induces a projection of $L(l, U)$ to U, which is still be denoted by $\mathcal{P}$, and the formula (2.20) also define a linear functional on $U^* \otimes L(l, U)$. Then (see [FZ] or [Li2]) $M(\ell, U)$ and $L(\ell, U)$ are weak modules for vertex operator algebra $V(\ell, \mathbb{C})$. Let $Y(\cdot, z)$ be the vertex operators defining the module structure on $L(\ell, \mathbb{C})$. It is clear that $Y(\cdot, z)$ is an intertwining operator of type $\begin{pmatrix} L(\ell, U) \\ V(\ell, \mathbb{C}) \, L(\ell, U) \end{pmatrix}$ (see [FHL] for the definition of intertwining operator). Let $\mathcal{Y}(\cdot, z)$ be the intertwining operator of type $\begin{pmatrix} L(\ell, U) \\ L(\ell, U) \, V(\ell, \mathbb{C}) \end{pmatrix}$ defined by $\mathcal{Y}(u, z)v = e^{zL(-1)}Y(v, -z)u$ (cf. [FHL]).

Lemma 2.9 *The $\tilde{\mathfrak{g}}$-module $L(\ell, U)$ is a weak module for the vertex operator algebra $L(\ell, 0)$ if and only if*

$$\langle u', \mathcal{Y}(u, z)v_{0,2} \rangle = 0 \quad \text{for any } u' \in U^*, u \in U(\mathfrak{g}) \subset L(\ell, U). \tag{2.22}$$

Proof. It is clear that the condition is necessary. Now we assume that (2.22) holds. Let J be the maximal submodule of $V(\ell, \mathbb{C})$ which intersects $\mathbb{C}$ trivially. Then $J = U(N_-)v_{0,2}$. From the definition of the bilinear form we get

$$\langle u', a\mathcal{Y}(u, z)w \rangle = 0 \quad \text{for } u' \in U^*, u \in L(\ell, U), a \in N_- U(N_-), w \in L(\ell, 0). \tag{2.23}$$

By using the commutator formula

$$[a(m), \mathcal{Y}(u, z)] = \sum_{j \geq 0} \binom{m}{j} \mathcal{Y}(a(j)u, z) z^{m-j} \tag{2.24}$$

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for $a \in \mathfrak{g}$, $m \in \mathbb{Z}$ and $u \in L(l, U)$ together with (2.22) we get

$$\langle u', \mathcal{Y}(u, z)J \rangle = 0 \quad \text{for any } u' \in U^*, u \in U(\mathfrak{g})U \subset L(\ell, U). \tag{2.25}$$

From the Jacobi identity for the vertex operators against the intertwining operator we have

$$\mathcal{Y}(a(n)u, z) = \sum_{j \geq 0} \binom{n}{j} a(n-j)\mathcal{Y}(u, z)z^j - (-1)^n \sum_{j \geq 0} \binom{n}{j} \mathcal{Y}(u, z)a(j)z^{n-j} \tag{2.26}$$

for $u \in L(l, U), a \in \mathfrak{g}$ and $n \in \mathbb{Z}$. Note that $L(l, U)$ is generated by U as $\tilde{\mathfrak{g}}$-module. Combining (2.23), (2.25) and (2.26) gives

$$\langle u', \mathcal{Y}(u, z)J \rangle = 0 \quad \text{for any } u' \in U^*, u \in L(\ell, U). \tag{2.27}$$

By the commutator formula (2.24) again, we obtain

$$\langle u', x\mathcal{Y}(u, z)J \rangle = 0 \quad \text{for any } u' \in U', x \in U(\tilde{\mathfrak{g}}), u \in L(\ell, U). \tag{2.28}$$

By the definition of $L(\ell, U)$ we have $\mathcal{Y}(v, z)u = 0$ for any $v \in J, u \in L(\ell, U)$. Thus, $\mathcal{Y}(\cdot, z)$ induces an intertwining operator of type $\begin{pmatrix} L(\ell, U) \\ L(\ell, U) \, L(\ell, 0) \end{pmatrix}$. This proves that $L(\ell, 0)$ is a weak module for $L(\ell, 0)$. $\quad \square$

Proposition 2.10 *The $L(\ell, U)$ is a weak $L(\ell, 0)$-module if and only if $f(h)U = 0$, where*

$$f(h) = \prod_{r=0}^{p-2} \prod_{s=0}^{q-1} (h - r + st).$$

Proof. Recall that $\mathcal{Y}(\cdot, z)$ is the corresponding nonzero intertwining operator of type $\begin{pmatrix} L(\ell, U) \\ L(\ell, U) \, V(\ell, \mathbb{C}) \end{pmatrix}$. For $n \in \mathbb{Z}, a \in \mathfrak{g}$ we define $\deg(x^n \otimes a) = n$. By (2.24) we obtain

$$\langle u', \mathcal{Y}(u, z)av \rangle = \langle u', z^{\deg a}\mathcal{Y}(\sigma P_1(a)u, z)v \rangle \tag{2.29}$$

for $u' \in U^*, u \in U(\mathfrak{g})U \subseteq L(\ell, U), a \in U(N_-), v \in L(\ell, 0)$. Let $a = F_2(1, 1)$. Then $v_{0,2} = a\mathbf{1}$. By Lemma 2.9 and (2.29) $L(l, U)$ is a weak $L(l, 0)$-module if and only if

$$\langle u', \mathcal{Y}(\sigma P_1(a)u, z)\mathbf{1} \rangle = 0. \tag{2.30}$$

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By Proposition 2.6, we have

$$P_1(a) = \prod_{r=1}^{p-1} \prod_{s=1}^{q-1} H_{-r-st} e^{p-1}. \tag{2.31}$$

Then from (2.8)

$$\sigma P_1(x) = (-1)^{p-1} e^{p-1} \prod_{r=1}^{p-1} \prod_{s=1}^{q-1} H_{r-1+st} = (-1)^{p-1} \prod_{r=1}^{p-1} \prod_{s=1}^{q-1} H_{-p+r+st} e^{p-1}. \tag{2.32}$$

Note that

$$\langle u', \mathcal{Y}(\sigma P_1(a) u, z) \mathbf{1} \rangle$$

$$= \langle u', e^{zL(-1)} (\sigma P_1(a)) u \rangle$$

$$= \langle u', (\sigma P_1(a)) u \rangle.$$

Thus $L(\ell, U)$ is a weak $L(\ell, 0)$-module if and only if

$$\langle u', \prod_{r=1}^{p-1} \prod_{s=1}^{q-1} H_{-p+r+st} e^{p-1} U(g) U \rangle = 0 \quad \text{for any } u' \in U^*. \tag{2.33}$$

From the grading restriction on the bilinear pair, the later is equivalent to

$$\left(\prod_{r=1}^{p-1} \prod_{s=1}^{q-1} H_{-p+r+st} e^{p-1} f^{p-1} \right) U = 0.$$

By (2.9) and the fact that $eU = 0$ we have

$$\prod_{r=1}^{p-1} \prod_{s=1}^{q-1} \prod_{i=1}^{p-1} H_{-p+r+st} H_{-i} U$$

$$= \prod_{r=1}^{p-1} \prod_{s=1}^{q-1} \prod_{i=1}^{p-1} (p-r-st)(h-p+r+1+st) i(h-i+1) U = 0. \tag{2.34}$$

Since $p - r - st \neq 0$ for any $1 \leq r \leq p-1, 1 \leq s \leq q-1$ (from Remark 2.3), we obtain

$$\prod_{r=1}^{p-1} \prod_{s=1}^{q-1} \prod_{i=1}^{p-1} (h-p+1+r+st)(h-i+1) U = 0. \tag{2.35}$$

Thus

$$\prod_{r=0}^{p-2} \prod_{s=0}^{q-1} (h-r+st) U = 0. \tag{2.36}$$

This finishes the proof.　□

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Corollary 2.11 *The highest weight $\hat{sl}_2$-module $L(\ell, j)$ is a weak $L(\ell, 0)$-module if and only if $j = r - st$ for $0 \leq r \leq p - 2, 0 \leq s \leq q - 1$. That is, $\hat{\mathfrak{g}}$-module $L(\ell, j)$ is a weak $L(\ell, 0)$-module if and only if j is admissible.*

Let j be an admissible weight so that $L(\ell, j)$ is a weak $L(\ell, 0)$-module. It follows from [FHL] that $L(\ell, 0)'$ is a weak $L(\ell, 0)$-module. But $(L(\ell, j)')' \neq L(\ell, j)$ because $L(\ell, j)$ has infinite-dimensional homogeneous subspaces in general. By using the well-known principal grading (cf. [K]), any $L(\ell, j) = \oplus_{m,n \in \mathbb{Z}} L(\ell, j)_{(m,n)}$ becomes a $\mathbb{Z}^2$-graded space such that each homogeneous subspace is finite-dimensional. Let $L(\ell, j)^c = \oplus_{(m,n) \in \mathbb{Z} \times \mathbb{Z}} L(\ell, j)^*_{m,n}$ be the restricted dual of $L(\ell, j)$ with respect to this $\mathbb{Z}^2$-grading. Then it is clear that $L(\ell, j)^c$ is an irreducible weak $L(\ell, 0)$-module satisfying $(L(\ell, j)^c)^c = L(\ell, j)$. But the lowest $L(0)$-weight subspace of $L(\ell, j)^c$ is a lowest weight $\mathfrak{g}$-module with $-j$ as its lowest weight. Then there is a non-trivial intertwining operator of type $\begin{pmatrix} L(\ell, 0) \\ L(\ell, j) L(\ell, j)^c \end{pmatrix}$ so that $L(\ell, j)$ and $L(\ell, j)^c$ are conjugate each other from the physical point of view.

Corollary 2.12 *Let j be an admissible weight. Then both $L(\ell, j)$ and $L(\ell, j)^c$ are irreducible weak $L(\ell, 0)$-modules.*

Remark 2.13 *If ℓ is not a nonnegative integer, there are also other types of irreducible weak $L(\ell, 0)$-modules. For a positive integral level ℓ, it was proved [DLiM2] that any weak module is completely reducible and any irreducible weak $L(\ell, 0)$-module is an irreducible integrable highest weight $\hat{\mathfrak{g}}$-module of level ℓ. This distinguishes $L(\ell, 0)$ for a positive integral level ℓ from all the rational levels.*

Remark 2.14 *It follows immediately from Propositions 2.10 and a complete reducibility theorem of Kac-Wakimoto (Theorem 4.1 of [KW2]) that any weak $L(\ell, 0)$-module M which is an $\hat{\mathfrak{g}}$-module of level ℓ from the category $\mathcal{O}$ is a direct sum of irreducible modules $L(\ell, j)$ with admissible weight j.*

Nest, we shall prove a completely reducibility theorem for a category much bigger than the category $\mathcal{O}$. Recall the following theorem from [KK]:

Theorem 2.15 *Let* $\lambda, \mu \in H^*$. *Then* $L(\mu)$ *is isomorphic to a subquotient module of* $M(\lambda)$ *iff the ordered pair* $\{\lambda, \mu\}$ *satisfies the following condition: There exists a sequence* $\beta_1, \cdots, \beta_k$ *of positive roots and a sequence* $n_1, \cdots, n_k$ *of positive integers such that*

(i) $\lambda - \sum_{i=1}^{k} n_i \beta_i = \mu$;

(ii) $2(\lambda + \rho - n_1\beta_1 - \cdots - n_{j-1}\beta_{j-1}, \beta_j) = n_j(\beta_j, \beta_j)$ *for* $1 \leq j \leq k$.

Lemma 2.16 *Let* λ, μ *be two distinct admissible weights. Then* $L(\mu)$ *is not isomorphic to any subquotient module of* $M(\lambda)$.

Proof. Otherwise, by Theorem 2.15 we have a sequence $\beta_1, \cdots, \beta_k$ of positive roots and a sequence $n_1, \cdots, n_k$ of positive integers satisfying (i)-(ii). From [DGK] each β_i is real. Then we obtain

$$\langle \mu + \rho, \beta_k \rangle = \langle \lambda + \rho - \sum_{i=1}^{k} n_i \beta_i, \beta_k \rangle = \frac{2(\lambda + \rho - \sum_{i=0}^{k} n_i \beta_i, \beta_k)}{(\beta_k, \beta_k)} = -n_k. \qquad (2.37)$$

This contradicts the admissibility of μ. $\square$

Lemma 2.17 *Let* M *be a weak* $L(\ell, 0)$-*module such that* M *is a highest weight* $\tilde{\mathfrak{g}}$-*module. Then* M *is irreducible.*

Proof. Let λ be the highest weight of M. If M contains a proper submodule W, there is a highest weight vector u in W of weight μ such that $\mu < \lambda$. Then both λ and μ are admissible by Corollary 2.11. This contradicts Lemma 2.16. Then M is irreducible. $\square$

Recall from [K] that $\hat{\omega}$ is the involutory antiautomorphism of $\hat{\mathfrak{g}}$, which is the negative Chevalley involution. Let M be a $\hat{\mathfrak{g}}$-module of level ℓ such that H local finitely acts on M with finite-dimensional generalized H-eigenspaces. We define [DGK] $M^{\hat{\omega}} = \oplus_{\lambda \in H^*} M_\lambda^*$ with the following action $(af)(u) = f(\hat{\omega}(a)u)$ for any $f \in M^{\hat{\omega}}, a \in \hat{\mathfrak{g}}, u \in M$. Then

$$(M^{\hat{\omega}})^{\hat{\omega}} \simeq M, \ L(\lambda)^{\hat{\omega}} \simeq L(\lambda) \quad \text{for any } \lambda \in H^*.$$

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Proposition 2.18 *Let* λ_1, λ_2 *be admissible weights of level* ℓ. *Then any short exact sequence*

$$0 \to L(\lambda_1) \to M \to L(\lambda_2) \to 0 \tag{2.38}$$

of weak $L(\ell, 0)$*-modules splits.*

Proof. First, since H semisimply acts on $L(\lambda_1)$ and $L(\lambda_2)$, H acts local finitely on M. Let $M = \oplus_{\lambda \in H^*} M_\lambda$ be the generalized H-eigenspace decomposition. Then the sequence (2.38) splits if and only if the following sequence splits:

$$0 \to L(\lambda_2) \to M^{\hat{\omega}} \to L(\lambda_1) \to 0. \tag{2.39}$$

Without losing generality we may assume that $\lambda_1 \not> \lambda_2$. Let $u \in M_{\lambda_2}$ such that $u \notin L(\lambda_1)$. Then $N_+ u \subseteq L(\lambda_1)$. If $N_+ u \neq 0$, there is a $\beta \in \mathbb{Z}_+ \alpha_0 + \mathbb{Z}_+ \alpha_1$ such that $\lambda_2 + \beta = \lambda_1$. This contradicts the assumption $\lambda_1 \not> \lambda_2$. Thus $N_+ u = 0$. Set $U = U(\mathfrak{g})u$. Let W be the submodule generated by U. Since $L(\ell, U)$ as a $\hat{\mathfrak{g}}$-module is isomorphic to some quotient module of W, $L(\ell, U)$ is a weak $L(\ell, 0)$-module. From Proposition 2.10, H semisimplely acts on U. Then u is a highest weight vector. By Lemma 2.17, W is irreducible. Then we obtain $M = W \oplus L(\lambda_1)$. That is, sequence (2.38) splits. $\square$.

From Proposition 2.18 we have

Corollary 2.19 *Let* $\lambda, \lambda_1, \cdots, \lambda_k$ *be admissible weights of level* ℓ. *Then any short exact sequence*

$$0 \to L(\lambda_1) \oplus \cdots \oplus L(\lambda_k) \to M \to L(\lambda) \to 0$$

of weak $L(\ell, 0)$*-modules splits.*

Theorem 2.20 *Let M be any weak $L(\ell, 0)$-module such that for any $u \in M$, there exists a positive integer k such that $(N_+)^k u = 0$. Then M is a direct sum of irreducible modules $L(\ell, j)$ with admissible weight j.*

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Proof. Set $\Omega(M) = \{m \in M | \mathfrak{g} \otimes t\mathbb{C}[t] \cdot m = 0\}$. Then the proof of Theorem 3.7 of [DLiM2] shows that $\Omega(M) \neq 0$. Since e is locally nilpotent on $\Omega(M)$ we conclude that there exist vectors $m \in M$ such that $N_+ m = 0$. From the proof of Proposition 2.18 we see that H acts semisimply on $U(\mathfrak{g})m$. Thus M contains a highest weight vector. It follows from Lemma 2.17 that M contains an irreducible weak $L(\ell, 0)$-module $L(\lambda)$. Let W be the sum of all irreducible weak $L(\ell, 0)$-submodules of M. We have to prove $M = W$. If $M \neq W$, there is a submodule E of M such that $W \subseteq E$, $E/W \simeq L(\lambda)$ for some admissible weight λ. Let $u + W$ be a highest weight vector of E/W. Since $\hat{\mathfrak{g}}$ is finitely generated,

$$N_+ u \subseteq L(\lambda_1) \oplus L(\lambda_2) \oplus \cdots \oplus L(\lambda_r)$$

for some $\lambda_1, \cdots, \lambda_r$. Set $W^o = L(\lambda_1) \oplus L(\lambda_2) \oplus \cdots \oplus L(\lambda_r)$. It follows from Corollary 2.19 that the submodule generated by u and W^o is completely reducible. Then $u \in W$. This contradicts the assumption of u. Thus $M = W$. This finishes the proof. $\square$

Remark 2.21 *From Corollary 2.11 and Proposition 2.20 the set of equivalence classes of irreducible $L(\ell, 0)$-modules consists of $L(\ell, j)$ with $j \in \mathbb{Z}, 0 \leq j \leq p - 2$ and any (ordinary) module is completely reducible.*

Remark 2.22 *In [Z], an associative algebra $A(V)$ was introduced for any vertex operator algebra V such that there is a natural one-to-one correspondence between the set of equivalence classes of irreducible $\mathbb{Z}_+$-graded weak V-modules and the set of equivalence classes of irreducible $A(V)$-modules. If ℓ is not a nonnegative integer, then $A(L(\ell, 0))$ has infinite-dimensional irreducible modules so that $A(L(\ell, 0))$ is infinite-dimensional. Therefore (from [DLiM1]), $L(\ell, 0)$ is not rational. Because Zhu's C_2-finiteness condition implies that $A(L(\ell, 0))$ is finite-dimensional, $L(\ell, 0)$ does not satisfy the C_2-finiteness condition.*

3 $\mathbb{Q}$-graded vertex operator algebras and the rationality of $(L(\ell, 0), \omega_z)$

If j is not a nonnegative integer, homogeneous spaces of $L(\ell, j)$ are infinite-dimensional so that the character $\mathrm{tr}_{L(\ell,j)} q^{L(0)}$ is not well-defined, where $c_\ell = \frac{3\ell}{\ell+2}$. In [KW1]-[KW2], the modified characters $\mathrm{tr}_{L(\ell,j)} q^{L(0)-\frac{1}{2}zh(0)-\frac{c_\ell}{24}}$ were considered, where z is a positive rational number less than 1. Noticing that $L(0) - \frac{1}{2}zh(0)$ could be considered as the degree-zero component of a Virasoro vector $\omega_z = \omega + \frac{1}{2}zh(-2)\mathbf{1}$ whose central charge is $c_{\ell,z} = c_\ell - 6\ell z^2$, we study $L(\ell, 0)$ with respect to the new Virasoro element ω_z in this section. We denote the new vertex operator algebra by ω_z $(L(\ell, 0), \omega_z)$. Note that ω_z $(L(\ell, 0), \omega_z)$ is $\mathbb{Q}$-graded instead of $\mathbb{Z}$-graded. This leads us to the study of $\mathbb{Q}$-graded vertex operator algebras. In particular We extend Zhu's $A(V)$-theory and Frenkel-Zhu's fusion rule formula to any $\mathbb{Q}$-graded vertex operator algebra. That is, we construct an associative algebra $A(V)$ for any $\mathbb{Q}$-graded VOA V and establish the one-to-one correspondence between the set of equivalence classes of irreducible $\mathbb{Q}_+$-graded weak V-modules and the set of equivalence classes of irreducible $A(V)$-modules. If V is $\frac{1}{2}\mathbb{Z}$-graded, our construction $A(V)$ and related results coincide with those for vertex operator superalgebra as developed in [KWa]. We also use complete reducibility Theorem 2.20 to show that $(L(\ell, 0), \omega_z)$ is rational.

A $\mathbb{Q}$-graded vertex operator algebra V satisfies all the axioms for a vertex operator algebra V except that V is $\mathbb{Q}$-graded by weights instead of $\mathbb{Z}$-graded. In particular, a $\mathbb{Q}$-graded vertex operator algebra is a generalized vertex operator algebra in the sense of [DL]. The definitions of weak module and ordinary module are as before. In the definition of $\mathbb{Z}_+$-graded module for a $\mathbb{Z}$-graded vertex operator algebra, replacing $\mathbb{Z}$ by $\mathbb{Q}$ gives a $\mathbb{Q}_+$-graded module for a $\mathbb{Q}$-graded vertex operator algebra.

Definition 3.1 *A $\mathbb{Q}$-graded vertex operator algebra V is called* rational *if any $\mathbb{Q}_+$-graded weak V-module is completely reducible.*

Let $V = \oplus_{\alpha \in \mathbb{Q}} V_\alpha$ be a $\mathbb{Q}$-graded vertex operator algebra. Then $V_\mathbb{Z} = \oplus_{n \in \mathbb{Z}} V_n$ is a $\mathbb{Z}$-graded (ordinary) vertex operator algebra. Just as in [FFR] and [Li2], one obtains a $\mathbb{Q}$-graded Lie algebra $G(V) = \oplus_{\alpha \in \mathbb{Q}} G(V)_\alpha$ as the quotient space of $\mathbb{C}[x, x^{-1}] \otimes V$ modulo $(\frac{d}{dx} \otimes 1 + 1 \otimes L(-1))(\mathbb{C}[x, x^{-1}] \otimes V)$. Here the Lie bracket is induced from

$$[x^m \otimes u, x^n \otimes] = \sum_{i=0}^{\infty} \binom{m}{i} x^{m+n-i} \otimes u_i v$$

for $u, v \in V$ and the degree of $x^n \otimes u + (\frac{d}{dx} \otimes 1 + 1 \otimes L(-1))(\mathbb{C}[x, x^{-1}] \otimes V)$ is $\mathrm{wt}u - n - 1$ for homogeneous u. Set

$$G(V)_\pm = \oplus_{\alpha > 0} G(V)_{\pm\alpha}. \tag{3.1}$$

Let U be any $G(V)_0$-module. Then we form the following induced module:

$$M(U) = U(G(V)) \otimes_{U(G(V)_0 \oplus G(V)_-)} U \tag{3.2}$$

where $G(V)_-$ acts trivially on U. Then $M(U)$ is a lower-truncated $\mathbb{Q}$-graded $G(V)$-module generated by the lowest-degree subspace U. Let U^* be the dual space of U and extend U^* to $M(U)$ by letting U^* annihilate $\oplus_{n>0} M(U)(n)$. We denote such a pair by $\langle u', v \rangle$ for $u' \in U^*$ and $v \in M(U)$. Set

$$I = \{ v \in M(U) | \langle u', av \rangle = 0 \ \text{ for any } u' \in U^*, a \in U(G(V)) \}. \tag{3.3}$$

Then it is clear that I is a $G(V)$-submodule of $M(U)$. Let $L(U)$ be the quotient module of $M(U)$ modulo I.

Let V be a $\mathbb{Q}$-graded vertex operator algebra. First we define a function ε for all homogeneous elements of V as follows: $\varepsilon(a) = 1$ if $\mathrm{wt}a \in \mathbb{Z}$, $\varepsilon(a) = 0$ if $\mathrm{wt}a \notin \mathbb{Z}$. For any homogeneous element $a \in V$, we define:

$$a * b = \varepsilon(a)\mathrm{Res}_x \frac{(1+x)^{[\mathrm{wt}a]}}{x} Y(a, x)b \quad \text{for any } b \in V, \tag{3.4}$$

where $[\cdot]$ denotes the greatest-integer function. Then extend "$*$" to a bilinear product on V. Let $O(V)$ be the subspace of V linearly spanned by

$$\mathrm{Res}_x \frac{(1+x)^{[\mathrm{wt}a]}}{x^{1+\varepsilon(a)}} Y(a,x)b \tag{3.5}$$

for any homogeneous element $a \in V$ and for any $b \in V$. Using $(1+x)^m = \sum_{i=0}^m \binom{m}{i} x^i$ one can prove

$$\mathrm{Res}_x \frac{(1+x)^{[a]+m}}{x^{n+1+\varepsilon(a)}} Y(a,x)b \in I \tag{3.6}$$

for $n \geq m \geq 0$. Let M be any weak V-module. Then we define

$$\Omega(M) = \{u \in M | a_m u = 0 \quad \text{for } a \in V, m > \mathrm{wt}a - 1\}.$$

Define o to be the linear map from V to $\mathrm{End}\Omega(M)$ such that $o(a) = \varepsilon(a)a_{[\mathrm{wt}a]-1}$ for any homogeneous element a of V. Generalizing Theorems 2.1.1 and 2.1.2 of Zhu we obtain

Theorem 3.2 *(a) The subspace $O(V)$ is a two-sided ideal of V with respect to the product "$*$" and $A(V) = V/O(V)$ is an associative algebra with identity $\mathbf{1} + O(V)$. Moreover, $\omega + O(V)$ lies in the center of $A(V)$.*

(b) For any weak V-module M, $\Omega(M)$ is an $A(V)$-module with a acts as $o(a)$.

The proof is the same as in the twisted case (see the proofs of Proposition 2.3 and Theorem 5.3 in [DLiM1]).

Similarly, for a weak V-module M, we define $O(M)$ to be the subspace of M linearly spanned by

$$\mathrm{Res}_x \frac{(1+x)^{[\mathrm{wt}a]}}{x^{1+\varepsilon(a)}} Y(a,x)u \tag{3.7}$$

for any homogeneous element $a \in V$ and for any $u \in M$. The following theorem is an analogue of Theorems 1.5.1 and 1.5.2 of [FZ] (also see [KWa] and [Li2]):

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Theorem 3.3 *(a) The quotient space $A(M) = M/O(M)$ is an $A(V)$-bimodule with the following left and right actions:*

$$a * u = \varepsilon(a)\text{Res}_x \frac{(1+x)^{[\text{wt}a]}}{x} Y(a, x)u, \tag{3.8}$$

$$u * a = \varepsilon(a)\text{Res}_x \frac{(1+x)^{[\text{wt}a]-1}}{x} Y(a, x)u \tag{3.9}$$

for any homogeneous $a \in V$ and for any $u \in M$.

(b) Let W_1, W_2, W_3 be irreducible V-modules and suppose V is rational[3]. Then there is a linear isomorphism from the space $\text{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} W_2(0), W_3(0))$ to the space of intertwining operators of type $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$.

Proof. Let $I(M)$ be the subspace of $O(M)$ linearly spanned by

$$\text{Res}_x \frac{(1+x)^{\text{wt}a}}{x^2} Y(a, x)u$$

for any homogeneous element $a \in V_{\mathbb{Z}}$ and for any $u \in M$. Then $A_{V_{\mathbb{Z}}}(M) = M/I(M)$ is the $A(V_{\mathbb{Z}})$-bimodule defined in [FZ]. Thus it is enough for us to prove that the subspace $O(M)/I(M)$ is a sub-bimodule of $A_{V_{\mathbb{Z}}}(M)$. Since the proof this is parallel to the proof of Theorem 3.2 we omit the proof.

The proof of (b) is similar to that for the $\mathbb{Z}$-graded vertex operator algebra as in [Li2]. □

By definition $A(V)$ is a quotient algebra of $A(V_{\mathbb{Z}})$. It is clear that $A(V)_{Lie}$ is a quotient algebra of $G(V)_0$. Then for any $A(V)$-module U, we may naturally view U as a $G(V)_0$-module.

Proposition 3.4 *For any $A(V)$-module U, $L(U)$ is a weak V-module.*

Proof. The proof is the same as in the ordinary case (see [Li2] and [Z]) or the twisted case (see the proof of Theorem 6.3 of [DLiM1]). □

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[3]It was pointed out in [Li2] that this condition is necessary and a proof was supplied

The following is a gneralization of Theorem 2.2.2 of [Z]. See [Li2] or [DLiM1] for a similar proof.

Theorem 3.5 *The functor Ω gives rise to a one-to-one correspondence between the set of equivalence classes of irreducible $\mathbb{Q}_+$-graded weak V-modules and the set of equivalence classes of irreducible $A(V)$-modules.*

As in the case of $\mathbb{Z}$-graded vertex operator algebra, we have (see the proof of Theorem 8.1 of [DLiM1]):

Proposition 3.6 *If V is rational, $A(V)$ is semisimple and any $\mathbb{Q}_+$-graded weak V-module is a direct sum of irreducible ordinary V-modules.*

Let V be a $\mathbb{Q}$-graded vertex operator algebra. Then it is clear that $\exp(2\pi i L(0))$ is an automorphism of V. Let $M = \oplus_{h\in\mathbb{C}}M_{(h)}$ be a V-module. Following [FHL], let $M' = \oplus_{h\in\mathbb{C}}M_h^*$ be the restricted dual of M and define

$$\langle Y(a,x)f, u\rangle = \langle f, Y(e^{xL(1)}(e^{\pi i}x^{-2})^{L(0)}a, x^{-1})u\rangle \tag{3.10}$$

for any $f \in M', a \in V, u \in M$. The following proposition is essentially proved in [Li3].

Proposition 3.7 *The pair $(M', Y(\cdot, x))$ gives rise to a σ^2-twisted V-module, where $\sigma = \exp(2\pi i L(0))$.*

Remark 3.8 *If V is $\frac{1}{2}\mathbb{Z}$-graded, then M' is a V-module because $\sigma^2 = id_V$. Therefore, we obtain a new functor from V-modules to V-modules. It is important to notice that the vertex operator algebra V may not be isomorphic to its own contragredient dual.*

Let V be a $\mathbb{Q}$-graded vertex operator algebra and let M be any weak V-module. Define $C_2(M)$ to be the subspace linearly spanned by $a_{-2}M$ for $a \in V_{\mathbb{Z}}$ and by $a_{-1}M$ for $a \in V_n, n \notin \mathbb{Z}$. Define bilinear products "$\cdot$" and "$\circ$" on V as follows: For $a \in V_m, b \in V_n$ we define $a\cdot b = a_{-1}b$ and $a\circ b = a_0b$ if $m, n \in \mathbb{Z}$, otherwise we define $a\cdot b = 0$ and $a\circ b = 0$.

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Lemma 3.9 *The defined subspace $C_2(V)$ is a two-sided ideal for both $(V, \cdot)$ and $(V, \circ)$.*

Proof. Let $a \in V_m, b \in V_n, c \in V_k$. If $m \notin \mathbb{Z}$ or $n + k \notin \mathbb{Z}$, by definition we have: $a \cdot b_{-r}c = 0$ and $a \circ b_{-r}c = 0$ for $r = 1$ or 2. If $m, n + k \in \mathbb{Z}$, we get

$$a_{-j}(b_{-r}c) = b_{-r}a_{-j}c + \sum_{i=0}^{\infty} \binom{-j}{i} (a_i b)_{-j-r-i}c$$

for $j = -1$ or 0. Then $a \cdot (b_{-r}c), a \circ (b_{-r}c) \in C_2(V)$. Then the proof is complete. $\square$

Set $A_2(M) = M/C_2(M)$. By Lemma 3.9 we obtain a quotient algebra $A_2(V) = V/C_2(V)$. Similarly to [Z] we have:

Proposition 3.10 *The quotient algebra $(A_2(V), \cdot)$ is a commutative associative algebra with the vacuum vector $\mathbf{1}$ as its identity and $(A_2(V), \circ)$ is a Lie algebra such that*

$$(a \cdot b) \circ c = a \cdot (b \circ c) + (a \circ c) \cdot b$$

for any $a, b, c \in A_2(V)$. Therefore $(A_2(V), \cdot, \circ)$ is a Poisson Lie algebra.

Definition 3.11 *If $A_2(V)$ is finite-dimensional, we say V is C_2-finite or V satisfies the C_2-finiteness condition. If V as a Virasoro algebra module is generated by primary vectors, we say that V satisfies the* Virasoro condition*. If V as a vertex operator algebra is generated by ω and all primary vectors, we say that V satisfies the* primary-field condition*.*

Remark 3.12 *It has been proved in [Z] that if V is a rational vertex operator algebra with integral weights satisfying the C_2-finiteness condition and the Virasoro condition, then the space linearly spanned by $\mathrm{tr}_M q^{L(0) - \frac{c}{24}}$, where M runs through all irreducible V-modules, is modular invariance. If one replaces the Virasoro condition by the primary-field condition, one can check that Zhu's theorem also holds.*

Recall the following proposition from [DLinM].

Proposition 3.13 *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra of rank r and let $h \in V$ satisfying the following conditions:*

$$L(n)h = \delta_{n,0}h, \ h_n h = \delta_{n,1}\lambda\mathbf{1} \ \text{ for } n \in \mathbb{Z}_+, \tag{3.11}$$

where λ is a complex number. Then $(V, Y, \mathbf{1}, \omega + h_{-2}\mathbf{1})$ is a vertex algebra of rank $r - 12\lambda$.

Now we go back to the vertex operator algebra $L(\ell, 0)$. For any $z \in \mathbb{Q}$, we set $\omega_z = \omega + \frac{1}{2}zh(-2)\mathbf{1}$. Then it follows from Proposition 4.1 of [DLinM] that ω_z is a new Virasoro vector of $L(\ell, 0)$ with a central charge $\frac{3\ell}{\ell+2} - 6\ell z^2$. Thus $L_z(0) = (\omega_z)_1 = L(0) - \frac{1}{2}zh(0)$ so that

$$[L_z(0), x^m \otimes h] = -m(x^m \otimes h); \tag{3.12}$$

$$[L_z(0), x^m \otimes e] = (-m - z)(x^m \otimes e); \tag{3.13}$$

$$[L_z(0), x^m \otimes f] = (-m + z)(x^m \otimes f) \tag{3.14}$$

for any $m \in \mathbb{Z}$. In general, V is $\mathbb{Q}$-graded by weights with respect to $L_z(0) = L(0) - \frac{1}{2}zh(0)$ instead of $\mathbb{Z}$-graded. Consequently, we obtain a $\mathbb{Q}$-grading for $\tilde{sl}_2$ satisfying the conditions:

$$\deg(x^n \otimes e) = -n - z; \ \deg(x^n \otimes f) = -n + z, \ \deg(x^n \otimes h) = -n \quad \text{for } n \in \mathbb{Z}. \tag{3.15}$$

For a positive integral level ℓ, all irreducible $L(\ell, 0)$-modules are integral modules. For a general rational level ℓ, admissible weight j may be non-integral. To make the graded spaces of $L(\ell, j)$ be finite-dimensional, we assume $z \in \mathbb{Q}, 0 < z < 1$.

Let $M = \oplus_{n \in \mathbb{Q}_+} M(n)$ be any $\mathbb{Q}_+$-graded weak $(L(\ell, 0), \omega_z)$-module. Since $x^n \otimes e, x^{n+1} \otimes f, x^{n+1} \otimes h$ for $n \in \mathbb{Z}_+$ have negative degrees with respect to the operator $L_z(0)$, it is clear that M satisfies the condition of Proposition 2.20 so that M is completely reducible. Then we obtain

Theorem 3.14 *The $\mathbb{Q}$-graded vertex operator algebra $(L(\ell, 0), \omega_z)$ is rational and all irreducible modules (up to equivalence) are $L(\ell, j)$ for the admissible weights j.*

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Remark 3.15 *It is easy to check that each eigenspace for $L_z(0)$ in $L(\ell, j)$ is finite-dimensional for admissible weight j. Thus $\mathrm{tr}_{L(\ell,j)} q^{L_z(0)} = \mathrm{tr}_{L(\ell,j)} q^{L(0) - \frac{1}{2} z h(0)}$ is well defined and is equal to $\mathrm{tr}_{L(\ell,j)^c} q^{L_z(0)}$. In fact they are convergent in upper half plane (see Section 5).*

Remark 3.16 *Since $L_z(n) = L(n) - \frac{1}{2}(n+1) z h(n)$, e, f are primary vectors in $(L(\ell, 0), \omega_z)$. Because $L(\ell, 0)$ as a vertex operator algebra is generated by e, f, $(L(\ell, 0), \omega_z)$ satisfies the primary-field condition.*

4   Fusion rules and C_2-finiteness of $(L(\ell, 0), \omega_z)$

The main goal of this section is to calculate the fusion rules and prove that $(L(\ell, 0), \omega_z)$ is C_2-finite. Throughout this section we assume that $\ell = -2 + \frac{p}{q}$, where p and q are coprime positive integers with $p \geq 2$ and that z is a fixed rational number satisfying $0 < z < 1$ (under which certain traces converge in some domain [KW1-2]).

Let M be any weak $V(\ell, \mathbb{C})$-module. Since

$$\mathrm{wt} h(-1) = 1, \ \mathrm{wt} e(-1) = 1 - z, \ \mathrm{wt} f(-1) = 1 + z, \tag{4.1}$$

we have

$$\mathrm{Res}_x \frac{(1+x)^{[\mathrm{wt} f]}}{x^m} Y(f, x) u = (f(-m) + f(1-m)) u; \tag{4.2}$$

$$\mathrm{Res}_x \frac{(1+x)^{[\mathrm{wt} e]}}{x^m} Y(e, x) u = e(-m) u; \tag{4.3}$$

$$\mathrm{Res}_x \frac{(1+x)^{\mathrm{wt} h}}{x^{m+1}} Y(h, x) u = (h(-m-1) + h(-m)) u \tag{4.4}$$

for any positive integer m and for $u \in M$. By definition all those elements in (4.2)-(4.4) are in $O(M)$.

Proposition 4.1 *Let M be any weak $V(\ell, \mathbb{C})$-module. Then the space $O(M)$ is spanned by the all the elements in (4.2)-(4.4).*

Proof. Let W be the subspace linearly spanned by the all elements in (4.2)-(4.4). Set

$$C = \mathbb{C}[x^{-1}](x^{-1} + 1) \otimes f + \mathbb{C}[x^{-1}]x^{-1} \otimes e + \mathbb{C}[x^{-1}](x^{-2} + x^{-1}) \otimes h. \qquad (4.5)$$

Then $W = C \cdot M$. Since $[h(-k), C] \subseteq C$ for any positive integer k, we get $h(-k)W \subseteq W$.

Let L be the linear span of homogeneous elements a of $V(\ell, \mathbb{C})$ such that for any positive integer n,

$$\mathrm{Res}_x \frac{(1 + x)^{[\mathrm{wt}a]}}{x^{n+\varepsilon(a)}} Y(a, x)M \subseteq W. \qquad (4.6)$$

We shall prove that L is equal to $V(\ell, \mathbb{C})$.

For any homogeneous element a of L and for any nonnegative integers $m \geq n$, we have

$$\mathrm{Res}_x \frac{(1 + x)^{[\mathrm{wt}a]+n}}{x^{m+1+\varepsilon(a)}} Y(a, x)M \subseteq W \qquad (4.7)$$

because $\dfrac{(1 + x)^{[\mathrm{wt}a]+n}}{x^{m+1+\varepsilon(a)}} = \sum_{i=0}^{\infty} \binom{n}{i} \dfrac{(1 + x)^{[\mathrm{wt}a]}}{x^{m-i+1+\varepsilon(a)}}.$

Let a be any homogeneous element of L and let k be any positive integer. Then for any $n \in \mathbb{N}, u \in M$, we have:

$$\mathrm{Res}_{z_2} \frac{(1 + z_2)^{[\mathrm{wt}h(-k)a]}}{z_2^{n+\varepsilon(h(-k)a)}} Y(h(-k)a, z_2)u$$

$$= \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} \frac{(1 + z_2)^{[\mathrm{wt}a]+k}}{z_2^{n+\varepsilon(a)}} z_0^{-k} Y(Y(h, z_0)a, z_2)u$$

$$= \mathrm{Res}_{z_1} \mathrm{Res}_{z_2} \frac{(1 + z_2)^{[\mathrm{wt}a]+k}}{z_2^{n+\varepsilon(a)}} (z_1 - z_2)^{-k} Y(h, z_1)Y(a, z_2)u$$

$$- \mathrm{Res}_{z_2} \mathrm{Res}_{z_1} \frac{(1 + z_2)^{[\mathrm{wt}a]+k}}{z_2^{n+\varepsilon(a)}} (-z_2 + z_1)^{-k} Y(a, z_2)Y(h, z_1)u$$

$$= \mathrm{Res}_{z_2} \sum_{i=0}^{\infty} \binom{-k}{i} (-z_2)^i \frac{(1 + z_2)^{[\mathrm{wt}a]+k}}{z_2^{n+\varepsilon(a)}} h(-k - i)Y(a, z_2)u$$

$$- \mathrm{Res}_{z_2} \sum_{i=0}^{\infty} \binom{-k}{i} (-1)^{k+i} \frac{(1 + z_2)^{[\mathrm{wt}a]+k}}{z_2^{n+k+i+\varepsilon(a)}} Y(a, z_2)h(i)u$$

$$\equiv \mathrm{Res}_{z_2} \sum_{i=0}^{\infty} \binom{-k}{i} z_2^i \frac{(1 + z_2)^{[\mathrm{wt}a]+k}}{z_2^{n+\varepsilon(a)}} h(-k)Y(a, z_2)u \mod W$$

$$= h(-k)\mathrm{Res}_{z_2} \frac{(1 + z_2)^{[\mathrm{wt}a]}}{z_2^{n+\varepsilon(a)}} Y(a, z_2)u$$

$$\equiv 0 \mod W. \qquad (4.8)$$

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Here we used the relation $h(-k-i)w \equiv (-1)^i h(-k)w \pmod{M}$ which follows from (4.4).

Therefore $h(-k)L \subseteq L$ for any $k \in \mathbb{N}$.

Similarly, we have

$$
\operatorname{Res}_{z_2} \frac{(1+z_2)^{[\mathrm{wt}e(-k)a]}}{z_2^{n+\varepsilon(e(-k)a)}} Y(e(-k)a, z_2)u
$$

$$
= \operatorname{Res}_{z_2} \sum_{i=0}^{\infty} \binom{-k}{i} (-z_2)^i \frac{(1+z_2)^{[\mathrm{wt}e(-k)a]}}{z_2^{n+\varepsilon(e(-k)a)}} e(-k-i)Y(a, z_2)u
$$

$$
-\operatorname{Res}_{z_2} \sum_{i=0}^{\infty} \binom{-k}{i} (-1)^{k+i} \frac{(1+z_2)^{[\mathrm{wt}e(-k)a]}}{z_2^{n+k+i+\varepsilon(e(-k)a)}} Y(a, z_2)e(i)u
$$

$$
\equiv -\operatorname{Res}_{z_2} \sum_{i=0}^{\infty} \binom{-k}{i} (-1)^{k+i} \frac{(1+z_2)^{[\mathrm{wt}e(-k)a]}}{z_2^{n+k+i+\varepsilon(e(-k)a)}} Y(a, z_2)e(i)u \quad \mathrm{mod}\ W. \qquad (4.9)
$$

If $\mathrm{wt}a \in \mathbb{Z}$, then $[\mathrm{wt}e(-k)a] = \mathrm{wt}a + k - 1$ and $\varepsilon(e(-k)a) = 0$. Then the last formula in (4.9) is equal to

$$
-\operatorname{Res}_{z_2} \sum_{i=0}^{\infty} \binom{-k}{i} (-1)^{k+i} \frac{(1+z_2)^{\mathrm{wt}a+k-1}}{z_2^{n+k+i}} Y(a, z_2)e(i)u
$$

which is in W by (4.7) as $\mathrm{wt}e = 1 - z < 1$ and $[\mathrm{wt}a] + k - 1 \le [\mathrm{wt}e(-k)a] \le [\mathrm{wt}a] + k$. A similar discussion using (4.7) shows that the last expression of (4.9) is also in W if $\mathrm{wt}a \notin \mathbb{Z}$. Thus $\operatorname{Res}_{z_2} \frac{(1+z_2)^{[\mathrm{wt}e(-k)a]}}{z_2^{n+\varepsilon(e(-k)a)}} Y(e(-k)a, z_2)u \in W$.

Analogously,

$$
\operatorname{Res}_{z_2} \frac{(1+z_2)^{[\mathrm{wt}f(-k)a]}}{z_2^{n+\varepsilon(f(-k)a)}} Y(f(-k)a, z_2)u
$$

$$
= \operatorname{Res}_{z_2} \sum_{i=0}^{\infty} \binom{-k}{i} (-z_2)^i \frac{(1+z_2)^{[\mathrm{wt}a+z]+k}}{z_2^{n+\varepsilon(f(-k)a)}} f(-k-i)Y(a, z_2)u
$$

$$
-\operatorname{Res}_{z_2} \sum_{i=0}^{\infty} \binom{-k}{i} (-1)^{k+i} \frac{(1+z_2)^{[\mathrm{wt}a+z]+k}}{z_2^{n+k+i+\varepsilon(f(-k)a)}} Y(a, z_2)f(i)u
$$

$$
\equiv \operatorname{Res}_{z_2} \sum_{i=0}^{\infty} \binom{-k}{i} (-1)^k z_2^i \frac{(1+z_2)^{[\mathrm{wt}a+z]+k}}{z_2^{n+\varepsilon(f(-k)a)}} f(0)Y(a, z_2)u
$$

$$
-\operatorname{Res}_{z_2} \sum_{i=0}^{\infty} \binom{-k}{i} (-1)^{k+i} \frac{(1+z_2)^{[\mathrm{wt}a+z]+k}}{z_2^{n+k+i+\varepsilon(f(-k)a)}} Y(a, z_2)f(i)u \quad \mathrm{mod}\ W
$$

$$
\equiv \operatorname{Res}_{z_2} (-1)^k \frac{(1+z_2)^{[\mathrm{wt}a+z]}}{z_2^{n+\varepsilon(f(-k)a)}} f(0)Y(a, z_2)u
$$

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$$-\text{Res}_{z_2}(-1)^k\frac{(1+z_2)^{[\text{wt}a+z]+k}}{z_2^{n+k+\varepsilon(f(-k)a)}}Y(a,z_2)f(0)u$$

$$-\text{Res}_{z_2}\sum_{i=1}^{\infty}\binom{-k}{i}(-1)^{k+i}\frac{(1+z_2)^{[\text{wt}a+z]+k}}{z_2^{n+k+i+\varepsilon(f(-k)a)}}Y(a,z_2)f(i)u \mod W$$

$$\equiv\ \text{Res}_{z_2}(-1)^k\frac{(1+z_2)^{[\text{wt}a+z]}}{z_2^{n+\varepsilon(f(-k)a)}}(f(0)Y(a,z_2)-Y(a,z_2)f(0))u$$

$$-\text{Res}_{z_2}\sum_{i=1}^{k}\binom{k}{i}(-1)^k\frac{(1+z_2)^{[\text{wt}a+z]}}{z_2^{n+i+\varepsilon(f(-k)a)}}Y(a,z_2)f(0)u$$

$$-\text{Res}_{z_2}\sum_{i=1}^{\infty}\binom{-k}{i}(-1)^{k+i}\frac{(1+z_2)^{[\text{wt}a+z]+k}}{z_2^{n+k+i+\varepsilon(f(-k)a)}}Y(a,z_2)f(i)u$$

$$\equiv\ \text{Res}_{z_2}(-1)^k\frac{(1+z_2)^{[\text{wt}f(0)a]}}{z_2^{n+\varepsilon(f(0)a)}}Y(f(0)a,z_2)u$$

$$-\text{Res}_{z_2}\sum_{i=1}^{k}\binom{k}{i}(-1)^k\frac{(1+z_2)^{[\text{wt}a+z]}}{z_2^{n+i+\varepsilon(f(-k)a)}}Y(a,z_2)f(0)u$$

$$-\text{Res}_{z_2}\sum_{i=1}^{\infty}\binom{-k}{i}(-1)^{k+i}\frac{(1+z_2)^{[\text{wt}a+z]+k}}{z_2^{n+k+i+\varepsilon(f(-k)a)}}Y(a,z_2)f(i)u \mod W. \qquad (4.10)$$

Since $\deg f(0)a = \deg a$, by the induction hypothesis, we have

$$\text{Res}_{z_2}\frac{(1+z_2)^{[\text{wt}f(0)a]}}{z_2^{n+\varepsilon(f(0)a)}}Y(f(0)a,z_2)u \in W.$$

Notice that $[\text{wt}a + z] = [\text{wt}a]$, or $[\text{wt}a] + 1$. If $[\text{wt}a + z] = [\text{wt}a]$, by (4.7) the last two terms in (4.10) are in W no matter what $\varepsilon(f(-k)a)$ is. If $[\text{wt}a + z] = [\text{wt}a] + 1$, then $\text{wt}a \notin \mathbb{Z}$ so that $\varepsilon(a) = 0$, it is clear that the last two terms in (4.10) are in W again by (4.7).

Since $\mathbf{1} \in L$ and $V(\ell,\mathbb{C}) = U(x^{-1}\mathbb{C}[x^{-1}]\otimes\mathfrak{g})\mathbf{1}$ we get $L = V(\ell,\mathbb{C})$. Thus $O(M) \subseteq W$. Therefore $O(M) = W$. $\square$

Proposition 4.2 *The associative algebra $A(V(\ell,\mathbb{C}))$ for $\mathbb{Q}$-graded vertex operator algebra $(V(\ell,\mathbb{C}),\omega_z)$ is isomorphic to the polynomial algebra $\mathbb{C}[x]$.*

Proof. Define a linear map ψ from $\mathbb{C}[x]$ to $A(V(\ell,\mathbb{C}))$ as follows

$$\psi(g(x)) = g(h(-1))\mathbf{1} + O(V(\ell,\mathbb{C})) \qquad (4.11)$$

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for $g(x) \in \mathbb{C}[x]$. Since $[h(-1), h(0)] = 0$ and $h(0)\mathbf{1} = 0$, we get $g(h(-1))\mathbf{1} = g(h(-1) + h(0))\mathbf{1}$ for any $g(x) \in \mathbb{C}[x]$. Since $\operatorname{wt} h(-1) = 1$ it follows from definition (3.4) that $h(-1) * u = (h(-1) + h(0))u$ for $u \in V(\ell, \mathbb{C})$. Thus ψ is an algebra homomorphism. Recall N_- and C from (2.3) and (4.5). Then $N_- = B \oplus \mathbb{C}f(0) \oplus \mathbb{C}h(-1)$. We have $U(N_-) = U(C)U(\mathbb{C}h(-1))U(\mathbb{C}f(0))$. By Proposition 4.1

$$O(V(\ell, \mathbb{C})) = CV(\ell, \mathbb{C}) = CU(N_-)\mathbf{1} \simeq CU(C)U(\mathbb{C}h(-1)).$$

Therefore ψ is an isomorphism. □

Proposition 4.3 *The $A(V)$-bimodule $A(M(\ell, j))$ is isomorphic to $\mathbb{C}[x, y]$ with the bi-action as follows:*

$$x * f(x, y) = (x + j - 2y\frac{\partial}{\partial y})f(x, y), \ f(x, y) * x = xf(x, y) \tag{4.12}$$

for any $f(x, y) \in \mathbb{C}[x, y]$.

Proof. Let v be a (nonzero) lowest weight vector of $M(\ell, j)$. Then as in the proof of Proposition 4.2 we have

$$O(M(\ell, j)) = CU(C)U(\mathbb{C}h(-1))U(\mathbb{C}f(0))v \simeq CU(C)U(\mathbb{C}h(-1))U(\mathbb{C}f(0)).$$

Then

$$A(M(\ell, j)) = \oplus_{m,n\in\mathbb{Z}_+}\mathbb{C}(h(-1)^m f(0)^n + O(M(\ell, j)).$$

By the definition of the left and right actions of $A(V(\ell, \mathbb{C}))$ on $A(M(\ell, j))$ in Theorem 3.3, we have

$$
\begin{aligned}
h(-1) * (h(-1)^m f(0)^n v) &= (h(-1) + h(0))h(-1)^m f(0)^n v \\
&= (h(-1) + j - 2n)h(-1)^m f(0)^n v \tag{4.13}
\end{aligned}
$$

and

$$(h(-1)^m f(0)^n v) * h(-1) = h(-1)(h(-1)^m f(0)^n v) = h(-1)^{m+1} f(0)^n v. \tag{4.14}$$

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The proposition follows immediately if we set $x = h(-1) + O(M(\ell, j)), y = f(0) + O(M(\ell, j))$. $\quad\square$

As a corollary of Propositions 2.10, 3.6 and Theorem 3.5 we obtain

Corollary 4.4 *The associative algebra $A(L(\ell, 0))$ is semisimple and isomorphic to the quotient algebra $\mathbb{C}[x]/\langle f(x)\rangle$ of the polynomial algebra $\mathbb{C}[x]$ in x, where*

$$f(x) = \prod_{r=0}^{p-2}\prod_{s=0}^{q-1}(x - r + st). \tag{4.15}$$

The following lemma is useful for calculating $A(L(\ell, j))$. The reader can refer to [FZ] for a proof.

Lemma 4.5 *(a) Let V be a vertex operator algebra and let M be a V-module with a submodule W. Set $\bar{M} = M/W$. Then as an $A(V)$-bimodule $A(\bar{M}) \simeq M/(O(M) + W)$.*

(b) If I is an ideal of V then $(I + O(V))/0(V)$ is a 2-sided ideal of $A(V)$ and $A(V/I)$ is isomorphic to $A(V)/((I + O(V))/O(V))$.

(c) If I is an ideal of V, and $I \cdot W \subset M$ ($I \cdot W$ means the linear span of elements $v_n w$ for $v \in I, n \in \mathbf{Z}$ and $w \in W$), then $I * A(M) \subset (W + O(M))/O(M)$, $A(M) * I \subset (W + O(M))/O(M)$, and $A(M)/((W + O(M)/O(M))$ is isomorphic to $A(W/M)$ as $A(V/I)$-bimodules.

Proposition 4.6 *Let $j = n - 1 - (k - 1)t$ be an admissible weight. Then the $A(L(\ell, 0))$-bimodule $A(L(\ell, j))$ is isomorphic to the quotient space of $\mathbb{C}[x, y]$ modulo the subspace*

$$\mathbb{C}[x, y]y^n + \mathbb{C}[x]f_{j,0}(x, y) + \mathbb{C}[x]f_{j,1}(x, y) + \cdots + \mathbb{C}[x]f_{j,n-1}(x, y)$$

where $f_{j,i}(x, y) = y^i \prod_{r=0}^{p-n-1}\prod_{s=0}^{q-k}(x - r - i + st)$. The left and right actions of $A(L(\ell, 0))$ on $A(L(\ell, j))$ are given by (4.12).

Proof. First, $M(\ell, j) \simeq U(N_-)$ as a vector space. Recall that $B_0 = \mathbb{C}(x^{-1} + 1) \otimes f + (x^{-2} + x^{-1})\mathbb{C}[x^{-1}] \otimes \mathfrak{g}$. Since $C = B_0 \oplus \mathbb{C}x^{-1} \otimes e$, by Proposition 4.1

$$O(M(\ell, j)) = CM(\ell, j) \simeq B_0 U(N_-) + e(-1)U(N_-).$$

Since B_0 is an ideal of N_-, $U(N_-)B_0 = B_0 U(N_-)$ is an ideal of $U(N_-)$. Set $L_0 = N_-/B_0$. Recall from Section 2 that $T_+ = e(-1) + B_0$ $T_- = f + B_0$ and $T_0 = h(-1) + B_0$. Then L_0 is a Lie algebra spanned by T_+, T_-, T_0 and isomorphic to $\mathfrak{g}$ (see (2.13)).

Recall from Proposition 2.4 that $v_{j,1}, v_{j,2}$ are the two singular vectors of $M(\ell, j)$. Then by Lemma 4.5 and Proposition 4.1 we have

$$A(L(\ell, j)) \simeq M(\ell, j)/(CM(\ell, j) + U(N_-)v_{j,1} + U(N_-)v_{j,2})$$

$$\simeq U(N_-)/(B_0 U(N_-) + e(-1)U(N_-) + U(N_-)F_1(n, k) + U(N_-)F_2(n, k)) \quad (4.16)$$

as $A(L(\ell, 0))$-bimodules. Note that $U(N_-)/B_0 U(N_-) \cong U(L_0)$. Thus

$$A(L(\ell, j)) \simeq U(L_0)/(U(L_0)P(F_1(n, k)) + U(L_0)P(F_2(n, k)) + T_+ U(L_0)). \quad (4.17)$$

For any nonnegative integers a, b, d, using Proposition 2.7, (2.14) and the fact that $G_\alpha = T_+ T_- - (\alpha + 1)T_0 + \alpha(\alpha + 1)$ we obtain

$$T_+^a T_0^b T_-^d P(F_1(n, k))$$

$$= T_+^a T_0^b T_-^d \left(\prod_{r=0}^{n-1} \prod_{s=1}^{k-1} G_{r+st}\right) T_-^n$$

$$= T_+^a \left(\prod_{r=0}^{n-1} \prod_{s=1}^{k-1} G_{r+d+st}\right) T_0^b T_-^{d+n}$$

$$= T_+^a \left(\prod_{r=0}^{n-1} \prod_{s=1}^{k-1}(T_+ T_- - (r + d + 1 + st)T_0 + (r + d + st)(r + d + 1 + st)\right) T_0^b T_-^{n+d}$$

$$\equiv T_+^a \left(\prod_{r=0}^{n-1} \prod_{s=1}^{k-1}(-r - d - 1 - st)(T_0 - r - d - st)\right) T_0^b T_-^{n+d} \mod T_+ U(L_0). \quad (4.18)$$

Noticing that $-r - d - 1 - st \neq 0$ for any $0 \leq r \leq n-1, 1 \leq s \leq k-1, d \in \mathbb{Z}_+$ we obtain

$$U(L_0)P(F_1(n, k)) + T_+ U(L_0)$$

$$= T_+ U(L_0) + \sum_{d=0}^{\infty} \mathbb{C}[T_0] \left(\prod_{r=0}^{n-1} \prod_{s=1}^{k-1}(T_0 - r - d - st)\right) T_-^{n+d}. \quad (4.19)$$

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Similarly, let a, b, d be any nonnegative integers. If $d < p - n$, we have

$$
\begin{aligned}
& T_+^a T_0^b T_-^d P(F_2(n, k)) \\
= \ & T_+^a T_0^b T_-^d T_+^{p-n} \prod_{r=1}^{p-n} \prod_{s=1}^{q-k} G_{p-n-r-st} \\
= \ & T_+^a T_0^b \left(\prod_{i=0}^{d-1} G_i \right) T_+^{p-n-d} \prod_{r=1}^{p-n} \prod_{s=1}^{q-k} G_{p-n-r-st} \\
= \ & T_+^a T_0^b T_+^{p-n-d} \prod_{r=1}^{p-n} \prod_{s=1}^{q-k} \prod_{i=0}^{d-1} G_{p-n-r-st} G_{i+p-n-d} \\
= \ & T_+^{a+p-n-d} (T_0 - 2(a+p-n-d))^b \prod_{r=1}^{p-n} \prod_{s=1}^{q-k} \prod_{i=0}^{d-1} G_{p-n-r-st} G_{i+p-n-d} \\
\equiv \ & 0 \quad \mathrm{mod} \ T_+ U(L_0).
\end{aligned}
\tag{4.20}
$$

If $d = m + p - n$ for some $m \in \mathbb{Z}_+$, we have

$$
\begin{aligned}
& T_+^a T_0^b T_-^d P(F_2(n, k)) \\
= \ & T_+^a T_0^b T_-^m \prod_{i=0}^{p-n-1} G_i \prod_{r=1}^{p-n} \prod_{s=1}^{q-k} G_{p-n-r-st} \\
= \ & T_+^a T_0^b T_-^m \prod_{r=1}^{p-n} \prod_{s=0}^{q-k} G_{p-n-r-st} \\
= \ & T_+^a \left(\prod_{r=1}^{p-n} \prod_{s=0}^{q-k} G_{p+m-n-r-st} \right) T_0^b T_-^m \\
= \ & T_+^a \left(\prod_{r=0}^{p-n-1} \prod_{s=0}^{q-k} G_{m+r-st} \right) T_0^b T_-^m \\
\equiv \ & T_+^a \left(\prod_{r=0}^{p-n-1} \prod_{s=0}^{q-k} (-m-r-1+st)(T_0 - m - r + st) \right) T_0^b T_-^m \quad \mathrm{mod} \ T_+ U(L_0)
\end{aligned}
\tag{4.21}
$$

Since $-r - m - 1 + st \neq 0$ for any $0 \le r \le p - n - 1, 0 \le s \le q - k$, we obtain

$$
\begin{aligned}
& U(L_0) P(F_2(n, k)) + T_+ U(L_0) \\
= \ & T_+ U(L_0) + \sum_{m=0}^{\infty} \mathbb{C}[T_0] \left(\prod_{r=0}^{p-n-1} \prod_{s=0}^{q-k} (T_0 - m - r + st) \right) T_-^m.
\end{aligned}
\tag{4.22}
$$

Thus

$$
U(L_0) P(F_1(n, k)) + U(L_0) P(F_2(n, k)) + T_+ U(L_0)
$$

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$$\subset \ T_+ U(L_0) + U(L_0) T_-^n + \sum_{i=0}^{n-1} \mathbb{C}[T_0] \left(\prod_{r=0}^{p-n-1} \prod_{s=0}^{q-k} (T_0 - i - r + st) \right) T_-^i.$$

On the other hand, since $r + d + st \neq m + r' - s't$ for any $0 \leq r \leq n - 1, 1 \leq s \leq k - 1, 0 \leq r' \leq p - n - 1, 0 \leq s' \leq q - k, d, m \in \mathbb{Z}_+$, $\prod_{r=0}^{n-1} \prod_{s=1}^{k-1} (x - r - d - st)$ and $\prod_{r=0}^{p-n-1} \prod_{s=0}^{q-k} (x - m - r + st)$ are relatively prime. Then we obtain

$$\mathbb{C}[T_0] T_-^{n+i} \subseteq U(L_0) P(F_1(n,k)) + U(L_0) P(F_2(n,k)) + T_+ U(L_0)$$

for any $i \in \mathbb{Z}_+$. This shows that

$$U(L_0) P(F_1(n,k)) + U(L_0) P(F_2(n,k)) + T_+ U(L_0)$$
$$\supset \ T_+ U(L_0) + U(L_0) T_-^n + \sum_{i=0}^{n-1} \mathbb{C}[T_0] \left(\prod_{r=0}^{p-n-1} \prod_{s=0}^{q-k} (T_0 - i - r + st) \right) T_-^i.$$

Set $x = T_0, y = T_-$. Then the proposition follows from Proposition 4.3 and Lemma 4.5.

$\square$

Theorem 4.7 *For admissible weights $j_i = n_i - 1 - (k_i - 1)t$ $(i = 1, 2)$, the fusion rules are given as follows:*

$$L(\ell, j_1) \times L(\ell, j_2) = \sum_{i=\max\{0, n_1 + n_2 - p\}}^{\min\{n_1 - 1, n_2 - 1\}} L(\ell, j_1 + j_2 - 2i) \tag{4.23}$$

if $0 \leq k_2 - 1 \leq q - k_1$, and $L(\ell, j_1) \times L(\ell, j_2) = 0$ otherwise.

Proof. For any admissible weight j, let $\mathbb{C}v_j$ be the one-dimensional module for Lie algebra $\mathbb{C}h$ such that $hv_j = jv_j$. Then $\mathbb{C}v_j$ is the lowest weight space of $L(\ell, j)$. By Theorem 3.4 we need to calculate the $A(L(\ell, 0))$-module $A(L(\ell, j_1)) \otimes_{A(L(\ell,0))} \mathbb{C}v_{j_2}$. Using Proposition 4.6 we get

$$A(L(\ell, j_1)) \otimes_{A(L(\ell,0))} \mathbb{C}v_{j_2} \simeq \mathbb{C}[x, y]/J$$

where J is the subspace of $\mathbb{C}[x, y]$ spanned by

$$\{x - j_2, \mathbb{C}[x, y]y^{n_1}, f_{j_1, i}(j_2, 1)\mathbb{C}[x]y^i, i = 0, 1, \cdots, n_1 - 1\} \tag{4.24}$$

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If j_2 does not satisfy the relation $0 \leq k_2 - 1 \leq q - k_1$, then

$$f_{j_1,i}(j_2, 1) = \prod_{r=0}^{p-n_1-1} \prod_{s=0}^{q-k_1} (j_2 - r - i + st) \neq 0$$

for $0 \leq i \leq n_1 - 1$. Thus $A(L(\ell, j_1)) \otimes_{A(L(\ell,0))} \mathbb{C} v_{j_2} = 0$ so that all the corresponding fusion rules are zero.

Suppose $0 \leq k_2 - 1 \leq q - k_1$. As before $\mathbb{C}[x] y^i = 0$ in $\mathbb{C}[x, y]/J$ if $f_{j_1,i}(j_2, 1) \neq 0$. Notice that $f_{j_1,i}(j_2, 1) = \prod_{r=0}^{p-n_1-1} \prod_{s=0}^{q-k_1} (j_2 - r - i + st) = 0$ if and only if $j_2 - r - i + st = 0$ for some $0 \leq r \leq p - n_1 - 1, 0 \leq s \leq q - k_1$. This implies that $0 \leq r + i \leq p - 2$. It follows from Remark 2.3 that $r + i = n_2 - 1$. That is, $n_1 + n_2 - p \leq i \leq n_2 - 1$. Therefore

$$\max\{0, n_1 + n_2 - p\} \leq i \leq \min\{n_1 - 1, n_2 - 1\}.$$

If $n_1 + n_2 - p \leq i \leq n_2 - 1$, then $\mathbb{C}[x] y^i$ is not zero in $\mathbb{C}[x, y]/J$. Thus

$$\mathbb{C}[x, y]/J \cong \oplus_{\max\{0, n_1+n_2-p\} \leq i \leq \min\{n_1-1, n_2-1\}} \mathbb{C} y^i.$$

From (4.12) we get $x * y^i = (j_2 + j_1 - 2i) y^i$, as required. $\quad \square$

Remark 4.8 *(a) Since $L_z(-1) = L(-1)$ the fusion rules among the admissible modules with respect two different operator algebra structure of $L(\ell, 0)$ are the same. Thus the fusion rules obatined in Theorem 4.7 are also those with respect to the old vertex operator algebra structure.*

(b) After changing the notations one immediately sees that our results agree with Bernard and Felder's results [BF] on fusion rules by using BRST cohomology.

(c) Suppose that ℓ is an integer. That is, $q = 1$ and $p = \ell + 2$. Since $j_i = n_i - 1$, we have $n_1 + n_2 - p = j_1 + j_2 - \ell$. Since $k_i = 1$ for any i, $0 \leq k_2 - 1 \leq q - k_1$ holds automatically. Then

$$L(\ell, j_1) \times L(\ell, j_2) = \sum_{i=\max\{0, n_1+n_2-p\}}^{\min\{n_1-1, n_2-1\}} L(\ell, j_1 + j_2 - 2i)$$

$$= \sum_{i=0, i \geq j_1+j_2-\ell}^{\min\{j_1,j_2\}} L(\ell, j_1 + j_2 - 2i)$$

$$= \sum_{j=|j_1-j_2|, j+j_1+j_2 \leq 2\ell}^{j_1+j_2} L(\ell, j). \tag{4.25}$$

This is exactly the well-known fusion formula (cf. [GW], [TK]).

Proposition 4.9 *Let M be any $V(\ell, \mathbb{C})$-module. Then*

$$C_2(M) = (\mathbb{C}x^{-1} \otimes e + \mathbb{C}x^{-1} \otimes f + x^{-2}\mathbb{C}[x^{-1}] \otimes \mathfrak{g})M.$$

Proof. Since $\mathrm{wt}\, h = 1, \mathrm{wt}\, e, \mathrm{wt}\, f \notin \mathbb{Z}$, by the definition of $C_2(M)$ we get

$$(\mathbb{C}x^{-1} \otimes e + \mathbb{C}x^{-1} \otimes f + x^{-2}\mathbb{C}[x^{-1}] \otimes \mathfrak{g})M \subseteq C_2(M). \tag{4.26}$$

Set $B_1 = \mathbb{C}x^{-1} \otimes e + \mathbb{C}x^{-1} \otimes f + x^{-2}\mathbb{C}[x^{-1}] \otimes \mathfrak{g}$. Let a be a homogeneous element of $V(\ell, \mathbb{C})$ such that

$$\mathrm{Res}_{z_2} z_2^{-n-\varepsilon(a)} Y(a, z_2)M \subseteq B_1 M \tag{4.27}$$

for any positive integer n. Then for any $k, n \in \mathbb{N}, u \in M, b \in \{e, f, h\}$, we have

$$\mathrm{Res}_{z_2} z_2^{-n-\varepsilon(b(-k)a)} Y(b(-k)a, z_2)u$$

$$= \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} z_2^{-n-\varepsilon(b(-k)a)} z_0^{-k} Y(Y(b, z_0)a, z_2)u$$

$$= \mathrm{Res}_{z_1} \mathrm{Res}_{z_2} z_2^{-n-\varepsilon(b(-k)a)} (z_1 - z_2)^{-k} Y(b, z_1)Y(a, z_2)u$$

$$- \mathrm{Res}_{z_1} \mathrm{Res}_{z_2} z_2^{-n-\varepsilon(b(-k)a)} (-z_2 + z_1)^{-k} Y(a, z_2)Y(b, z_1)u$$

$$= \mathrm{Res}_{z_2} \sum_{i=0}^{\infty} \binom{-k}{i} (-z_2)^i z_2^{-n-\varepsilon(b(-k)a)} b(-k-i) Y(a, z_2)u$$

$$- \mathrm{Res}_{z_2} \sum_{i=0}^{\infty} \binom{-k}{i} (-1)^{k+i} z_2^{-n-k-i-\varepsilon(b(-k)a)} Y(a, z_2)b(i)u$$

$$\equiv \mathrm{Res}_{z_2} z_2^{-n-\varepsilon(b(-k)a)} b(-k)Y(a, z_2)u \mod W$$

$$\equiv 0 \mod B_1 W. \tag{4.28}$$

Clearly (4.27) holds for $a = \mathbf{1}$. Note that $V(\ell, 0) = U(x^{-1}\mathbb{C}[x] \otimes \mathfrak{g})\mathbf{1}$. It follows from (4.28) that (4.27) holds for all $a \in V(\ell, \mathbb{C})$. The proof is complete. □

Theorem 4.10 *The commutative associative algebra $A_2(L(\ell, 0), \omega_z)$ is isomorphic to the quotient algebra $\mathbb{C}[x]/\langle x^{(p-1)q} \rangle$. Consequently, $(L(\ell, 0), \omega_z)$ is C_2-finite.*

Proof. First, notice that the Verma module $M(\ell, 0)$ is linearly isomorphic to $U(N_-)$. Recall from Section 2 that $B_2 = \mathbb{C}x^{-1} \otimes f + x^{-2}\mathbb{C}[x^{-1}] \otimes \mathfrak{g}$ is an ideal of N_-, $L_2 = N_-/B_2$, is the corresponding quotient Lie algebra spanned by $\bar{e} = e(-1) + B_2, \bar{f} = f(0) + B_2, \bar{h} = h(-1) + B_2$ and with the commutation relations

$$[\bar{e}(-1), \bar{f}(0)] = \bar{h}(-1), [\bar{h}(-1), \bar{e}(-1)] = [\bar{h}(-1), \bar{f}(0)] = 0. \tag{4.29}$$

By Proposition 4.9, we get

$$C_2(M(\ell, 0)) = B_2 M(\ell, 0) + e(-1)M(\ell, 0) \simeq B_2 U(N_-) + e(-1)U(N_-).$$

One easily verifies that

$$A_2(L(\ell, 0)) \simeq U(L_2)/(\bar{e}U(L_2) + U(L_2)\bar{f} + U(L_2)P_2(F_2(1, 1))).$$

For any $a, b, m \in \mathbb{Z}_+$ and $m \geq p - 1$ we obtain from Proposition 2.8 that

$$
\begin{aligned}
& \bar{e}^a \bar{h}^b \bar{f}^m P_2(F_2(1, 1)) \\
=\ & \bar{e}^a \bar{h}^b \bar{f}^m \bar{e}^{p-1} \prod_{r=1}^{p-1} \prod_{s=1}^{q-1} \bar{H}_{p-1-r-st} \\
=\ & \bar{e}^a \bar{h}^b \bar{f}^{m-p+1} \prod_{i=0}^{p-2} \bar{H}_i \prod_{r=1}^{p-1} \prod_{s=1}^{q-1} \bar{H}_{p-1-r-st} \\
=\ & \bar{e}^a \bar{h}^b \left(\prod_{i=0}^{p-2} \prod_{r=1}^{p-1} \prod_{s=1}^{q-1} \bar{H}_{m-r-st} \bar{H}_{m-p+1+i} \right) \bar{f}^m \\
=\ & \bar{e}^a \bar{h}^b \left(\prod_{r=0}^{p-2} \prod_{s=0}^{q-1} \bar{H}_{r+m-p+1-st} \right) \bar{f}^{m-p+1}. \tag{4.30}
\end{aligned}
$$

Here we used the relations $\bar{H}_\alpha \bar{e} = \bar{e}\bar{H}_{\alpha+1}$, $\bar{H}_\alpha \bar{f} = \bar{f}\bar{H}_{\alpha+1}$ and $\bar{f}^s \bar{e}^s = \bar{H}_0 \cdots \bar{H}_{s-1}$ which follows from the definition of $\bar{H}_\alpha$ and the commutator relations (4.29). Thus if $a > 0$ or $m > p - 1$, then $\bar{e}^a \bar{h}^b \bar{f}^{p-1+m} P_2(F_2(1, 1)) \in \bar{e}U(L_2) + U(L_2)\bar{f}$.

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If $a = 0$ and $m = p - 1$ we have

$$\bar{h}^b \bar{f}^{p-1} P_2(F_2(1, 1))$$

$$= \bar{h}^b \prod_{r=0}^{p-2} \prod_{s=0}^{q-1} \bar{H}_{r-st}$$

$$\equiv \bar{h}^b \prod_{r=0}^{p-2} \prod_{s=0}^{q-1} (-r - 1 + st)\bar{h} \quad \text{mod } \bar{e}U(L_2). \qquad (4.31)$$

Similarly, if $m < p - 1$, for any $a, b \in \mathbb{Z}_+$ we get

$$\bar{e}^a \bar{h}^b \bar{f}^m P_2(F_2(1, 1)) \in \bar{e}U(L_2).$$

Since $-r - 1 + st \neq 0$ for any $1 \leq r \leq p - 1, 0 \leq s \leq q - 1$, we obtain

$$\bar{e}U(L_2) + U(L_2)\bar{f} + U(L_2)P_2(F_2(1, 1)) = \bar{e}U(L_2) + U(\bar{h})\bar{f} + U(L_2)\bar{h}^{(p-1)q}.$$

Then the theorem follows if we set $x = \bar{h}$. $\quad \square$

5 Modular invariance property

In this section we study modular invariance property of the space linearly spanned by all characters $tr_{L(\ell,j)}e^{2\pi i\tau(L(0)-\frac{1}{2}zh(0)-\frac{1}{24}(\frac{3\ell}{\ell+2}-6\ell z^2))}$, where $\text{im}\tau > 0, z \in \mathbb{Q}, 0 < z < 1$. In this section we shall first find a modular transformation formula for the modified characters for admissible modules.

Following [K] or [KW1]-[KW2], for $m, n \in \mathbb{Z}, m > 0$ we define

$$\theta_{n,m}(\tau, z) = \sum_{j \in \mathbb{Z}+\frac{n}{2m}} e^{2m\pi i\tau(j^2+jz)}, \quad z \in \mathbb{C}. \qquad (5.1)$$

Set

$$\Theta_{n,m}(\tau) = \sum_{j \in \mathbb{Z}+\frac{n}{2m}} e^{2m\pi i\tau j^2}. \qquad (5.2)$$

Then

$$\theta_{n,m}(\tau, z) = e^{2m\pi i\tau(-\frac{1}{4}z^2)} \sum_{j \in \mathbb{Z}+\frac{n}{2m}} e^{2m\pi i\tau(j+\frac{1}{2}z)^2}$$

$$= e^{-\frac{1}{2}mz^2\pi i\tau} \sum_{j \in \mathbb{Z}+\frac{n+mz}{2m}} e^{2m\pi i\tau j^2}. \qquad (5.3)$$

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Suppose that $z = \frac{v}{u}$ is a rational number with $u > 0$. Then

$$
\begin{aligned}
\theta_{n,m}(\tau, z) &= e^{-\frac{1}{2}mz^2\pi i\tau} \sum_{j \in \mathbb{Z} + \frac{nu+mv}{2mu}} e^{2m\pi i\tau j^2} \\
&= e^{-\frac{1}{2}mz^2\pi i\tau} \Theta_{nu+mv,mu}\left(\frac{\tau}{u}\right).
\end{aligned}
\tag{5.4}
$$

As in Section 4, we let $\ell = -2 + \frac{p}{q}$ be a fixed admissible level, where $p \geq 2, q$ are relatively prime positive integers. Let P_ℓ be the set of all admissible weights (mod $\mathbb{C}\delta$) of level ℓ. Then

$$
P_\ell = \{j = n - kt \,|\, n, k \in \mathbb{Z}_+, n \leq p - 2, k \leq q - 1\}.
$$

Set $c_\ell = \frac{3\ell}{\ell+2}$. For any rational number z, we set $c_{\ell,z} = c_\ell - 6\ell z^2$. In Section 4 we have studied the vertex operator algebra or chiral algebra $L(\ell, 0)$ under a different Virasoro vector ω_z which has a central charge $c_{\ell,z}$. That is, the rank of $(L(\ell, 0), \omega_z)$ is $c_{\ell,z}$. With this motivation we define the following characters

$$
\chi_j(\tau, z) := \mathrm{tr}_{L(\ell,j)} e^{2\pi i\tau(L_z(0) - \frac{1}{24}c_{\ell,z})} = \mathrm{tr}_{L(\ell,j)} e^{2\pi i\tau(L(0) - \frac{1}{2}zh(0) - \frac{1}{24}c_{\ell,z})}.
\tag{5.5}
$$

For an admissible weight $j = n - kt \in P_\ell$, set

$$
a = pq, \; b_j^\pm = q(\pm(n+1) - kt).
$$

Now we restrict z to be a positive rational number less than 1.

Remark 5.1 *In [KW1]-[KW2], the following defined character has been considered:*

$$
\bar{\chi}_j(\tau, z) = \mathrm{tr}_{L(\ell,j)} e^{2\pi i\tau(L(0) - \frac{1}{2}zh(0) - \frac{1}{24}c_\ell)},
\tag{5.6}
$$

and it was proved that

$$
\bar{\chi}_j(\tau, z) = \frac{\theta_{b_j^+,a}(\tau, q^{-1}z) - \theta_{b_j^-,a}(\tau, q^{-1}z)}{\theta_{1,2}(\tau, z) - \theta_{-1,2}(\tau, z)}.
\tag{5.7}
$$

Using KW's character formula we obtain

$$\chi_j(\tau, z)$$

$$= e^{\frac{1}{2}\ell z^2 \pi i \tau} \bar{\chi}_j(\tau, z)$$

$$= e^{\frac{1}{2}\ell z^2 \pi i \tau} \frac{\theta_{b_j^+, a}(\tau, q^{-1}z) - \theta_{b_j^-, a}(\tau, q^{-1}z)}{\theta_{1,2}(\tau, z) - \theta_{-1,2}(\tau, z)}$$

$$= e^{\frac{1}{2}\ell z^2 \pi i \tau} e^{-\frac{1}{2}aq^{-2}z^2 \pi i \tau} e^{z^2 \pi i \tau} \frac{\Theta_{qub_j^+ + av, aqu}\left(\frac{\tau}{qu}\right) - \Theta_{b_j^- qu + av, aqu}\left(\frac{\tau}{qu}\right)}{\Theta_{u+2v, 2u}\left(\frac{\tau}{u}\right) - \Theta_{-u+2v, 2u}\left(\frac{\tau}{u}\right)}$$

$$= e^{\frac{1}{2}z^2 \pi i \tau (\ell + 2 - aq^{-2})} \frac{\Theta_{qub_j^+ + av, aqu}\left(\frac{\tau}{qu}\right) - \Theta_{uqb_j^- + av, aqu}\left(\frac{\tau}{qu}\right)}{\Theta_{u+2v, 2u}\left(\frac{\tau}{u}\right) - \Theta_{-u+2v, 2u}\left(\frac{\tau}{u}\right)}$$

$$= \frac{\Theta_{uqb_j^+ + av, aqu}\left(\frac{\tau}{qu}\right) - \Theta_{qub_j^- + av, aqu}\left(\frac{\tau}{qu}\right)}{\Theta_{u+2v, 2u}\left(\frac{\tau}{u}\right) - \Theta_{-u+2v, 2u}\left(\frac{\tau}{u}\right)}. \tag{5.8}$$

Then χ_j is a modular function with $c_{\ell,z}$ as the modular anomaly rather than c_ℓ.

Remark 5.2 *In [KW1] the following transformation law was given:*

$$\bar{\chi}_j(-\tau^{-1}, \tau z) = \frac{1}{2i}\sqrt{\frac{2}{a}} \sum_{j' \in P_\ell} \left(e^{-i\pi b_+ b'_-/a} - e^{-i\pi b_+ b'_+/a}\right) \bar{\chi}_{j'}(\tau, z). \tag{5.9}$$

Later in [KW2], a correction was made by adding the factor $e^{\frac{1}{2}\ell z^2 \pi i \tau}$ on the right-hand side of (5.9). That is,

$$\bar{\chi}_j(-\tau^{-1}, \tau z) = \frac{1}{2i}\sqrt{\frac{2}{a}} e^{\frac{1}{2}\ell z^2 \pi i \tau} \sum_{j' \in P_\ell} \left(e^{-i\pi b_+ b'_-/a} - e^{-i\pi b_+ b'_+/a}\right) \bar{\chi}_{j'}(\tau, z). \tag{5.10}$$

Based on the modular transformation law ((5.9) without the factor $e^{\frac{1}{2}\ell z^2 \pi i \tau}$), the fusion rules have been calculated in [KS] and [MW] by using Verlinde formula [V]. Unfortunately, some of them are negative. On the other hand, the correct formula (5.10) can not be used to compute the fusion because the coefficients in (5.10) involve the variable τ. This puzzles both mathematicians and physicists.

For a $\mathbb{Z}$-graded rational vertex operator algebra satisfying C_2 condition and the Virasoro condition it is proved in [Z] that the space spanned by $\text{tr}_M q^{L(0) - \frac{c}{24}}$ for all irreducible

modules M modular invariant. If the Virasoro condition is replaced by the primary field condition (cf. Remark 3.16) one still has the modular invariance of the space by modifying Zhu's proof. Now we have a $\mathbb{Q}$-graded rational vertex operator algebra $(L(\ell, 0), \omega_z)$ satisfying C_2 condition (see Theorem 4.10) and primary field condition (see Remark 3.16). Unfortunately Zhu's modular invariance theorem [Z] does not apply to Q-graded vertex operator algebra. This raise a question: Is the space linearly spanned by $\chi_j(\tau, z)$ modular invariant under the transformation $\tau \mapsto -\tau^{-1}$ with z being fixed? This question will be discussed in our coming paper [DLiM3].

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